

# Order in Implication Zroupoids

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## Abstract

The variety **I** of implication zroupoids (using a binary operation  $\rightarrow$  and a constant 0) was defined and investigated by Sankappanavar in [7], as a generalization of De Morgan algebras. Also, in [7], several new subvarieties of **I** were introduced, including the subvariety **I**<sub>2,0</sub>, defined by the identity:  $x'' \approx x$ , which plays a crucial role in this paper. Some more new subvarieties of **I** are studied in [3] that includes the subvariety **SL** of semilattices with a least element 0; and an explicit description of semisimple subvarieties of **I** is given in [5].

It is a well known fact that there is a partial order (denote it by  $\sqsubseteq$ ) induced by the operation  $\wedge$ , both in the variety **SL** of semilattices with a least element and in the variety **DM** of De Morgan algebras. As both **SL** and **DM** are subvarieties of **I** and the definition of partial order can be expressed in terms of the implication and the constant, it is but natural to ask whether the relation  $\sqsubseteq$  on **I** is actually a partial order in some (larger) subvariety of **I** that includes both **SL** and **DM**.

The purpose of the present paper is two-fold: Firstly, a complete answer is given to the above mentioned problem. Indeed, our first main theorem shows that the variety **I**<sub>2,0</sub> is a maximal subvariety of **I** with respect to the property that the relation  $\sqsubseteq$  is a partial order on its members.. In view of this result, one is then naturally led to consider the problem of determining the number of non-isomorphic algebras in **I**<sub>2,0</sub> that can be defined on an  $n$ -element chain (herein called **I**<sub>2,0</sub>-chains),  $n$  being a natural number. Secondly, we answer this problem in our second main theorem which says that, for each  $n \in \mathbb{N}$ , there are exactly  $n$  nonisomorphic **I**<sub>2,0</sub>-chains of size  $n$ .

## 1 Introduction

The widely known fact that Boolean algebras can be defined using only implication and a constant was extended to De Morgan algebras in [7]. The crucial role played by a certain identity, called (I), led Sankappanavar, in 2012, to define and investigate, the variety **I** of implication zroupoids (I-zroupoids) generalizing De Morgan algebras. Also, in [7], he introduced several new subvarieties of **I** and found some relationships among those subvarieties. One of the subvarieties of **I**, called **I**<sub>2,0</sub>, defined by the identity:  $x'' \approx x$  and studied in [7], plays a crucial role in this paper. In [3], we introduce several more new subvarieties of **I**, including the subvariety **SL** which is term-equivalent to the (well known) variety of  $\vee$ -semilattices with a least element 0, and describe further relationships among the subvarities of **I**. An explicit description of semisimple subvarieties of **I** is given in [5].

It is also a well known fact that there is a partial order induced by the operation  $\wedge$ , both in the variety **SL** of semilattices with a least element and in the variety **DM** of De Morgan algebras. As both **SL** and **DM** are subvarieties of **I** and the definition of partial order can be expressed in terms of the implication and constant, it is but natural to ask whether the relation  $\sqsubseteq$  (now defined) on **I** is actually a partial order in some (larger) subvariety of **I** that includes both **SL** and **DM**.

The purpose of the present paper is two-fold: Firstly, a complete answer is given to the above mentioned problem. Indeed, our first main theorem shows that the variety  $\mathbf{I}_{2,0}$  is a maximal subvariety of  $\mathbf{I}$  with respect to the property that the relation  $\sqsubseteq$ , defined by:

$$x \sqsubseteq y \text{ if and only if } (x \rightarrow y)' = x, \text{ for } x, y \in \mathbf{A} \text{ and } \mathbf{A} \in \mathbf{I},$$

is a partial order. In view of this result, one is then naturally led to consider the problem of determining the number of non-isomorphic algebras in  $\mathbf{I}_{2,0}$  ( $\mathbf{I}_{2,0}$ -chains) that can be defined on an  $n$ -element set,  $n$  being a natural number. Secondly, we answer this problem in our second main result which says that, for each  $n \in \mathbb{N}$ , there are exactly  $n$  nonisomorphic  $\mathbf{I}_{2,0}$ -chains of size  $n$ .

## 2 Preliminaries

In this section we recall some definitions and results from [3], [5] and [7] that will be needed for this paper. Basic references are [1] and [2].

**Definition 2.1** [7] A groupoid with zero (*zroupoid*, for short) is an algebra  $\mathbf{A} = \langle A, \rightarrow, 0 \rangle$ , where  $\rightarrow$  is a binary operation and 0 is a constant. A zroupoid  $\mathbf{A} = \langle A, \rightarrow, 0 \rangle$  is an *implication zroupoid* (I-zroupoid, for short) if the following identities hold in  $\mathbf{A}$ , where  $x' := x \rightarrow 0$ :

$$(I) \quad (x \rightarrow y) \rightarrow z \approx [(z' \rightarrow x) \rightarrow (y \rightarrow z)']'$$

$$(I_0) \quad 0'' \approx 0.$$

The variety of I-zroupoids is denoted by  $\mathbf{I}$ .

In this paper we use the characterizations of De Morgan algebras, Kleene algebras and Boolean algebras (see [7]), and semilattices with least element 0 (see [3]), as definitions.

**Definition 2.2** An *implication zroupoid*  $\mathbf{A} = \langle A, \rightarrow, 0 \rangle$  is a *De Morgan algebra* (**DM-algebra**, for short) if  $\mathbf{A}$  satisfies the axiom:

$$(DM) \quad (x \rightarrow y) \rightarrow x \approx x.$$

A **DM-algebra**  $\mathbf{A} = \langle A, \rightarrow, 0 \rangle$  is a *Kleene algebra* (**KL-algebra**, for short) if  $\mathbf{A}$  satisfies the axiom:

$$(KL_1) \quad (x \rightarrow x) \rightarrow (y \rightarrow y)' \approx x \rightarrow x$$

or, equivalently,

$$(KL_2) \quad (y \rightarrow y) \rightarrow (x \rightarrow x) \approx x \rightarrow x.$$

A **DM-algebra**  $\mathbf{A} = \langle A, \rightarrow, 0 \rangle$  is a *Boolean algebra* (**BA-algebra**, for short) if  $\mathbf{A}$  satisfies the axiom:

$$(BA) \quad x \rightarrow x \approx 0'.$$

An *implication zroupoid*  $\mathbf{A} = \langle A, \rightarrow, 0 \rangle$  is a *semilattice with 0* (**SL-algebra**, for short) if  $\mathbf{A}$  satisfies the axioms:

$$(SM1) \quad x' \approx x$$

$$(SM2) \quad x \rightarrow y \approx y \rightarrow x. \text{ (Commutativity).}$$

We denote by **DM**, **KL**, **BA** and **SL**, respectively, the variety of **DM**-algebras, **KL**-algebras, **BA**-algebras, and **SL**-algebras.

We recall from [7] the definition of another subvariety of **I**, namely  $\mathbf{I}_{2,0}$ , which plays a fundamental role in this paper.

**Definition 2.3**  $\mathbf{I}_{2,0}$  denotes the subvariety of **I** defined by the identity:

$$x'' \approx x.$$

We note that **DM**, **KL**, **BA** and **SL** are all subvarieties of  $\mathbf{I}_{2,0}$  (see [7] and [3]).

**Lemma 2.4** [7, Theorem 8.15] *Let  $\mathbf{A}$  be an  $I$ -zroupoid. Then the following are equivalent:*

- (a)  $0' \rightarrow x \approx x$
- (b)  $x'' \approx x$
- (c)  $(x \rightarrow x')' \approx x$
- (d)  $x' \rightarrow x \approx x$ .

**Lemma 2.5** [7] *Let  $\mathbf{A} \in \mathbf{I}_{2,0}$ . Then*

- (a)  $x' \rightarrow 0' \approx 0 \rightarrow x$
- (b)  $0 \rightarrow x' \approx x \rightarrow 0'$ .

Several identities true in  $\mathbf{I}_{2,0}$  are given in [3], [5] and [7]. Some of those that are needed for this paper are listed in the next lemma, which also presents some new identities of  $\mathbf{I}_{2,0}$  that will be useful later in this paper. The proof of the lemma is given in the Appendix.

**Lemma 2.6** *Let  $\mathbf{A} \in \mathbf{I}_{2,0}$ . Then  $\mathbf{A}$  satisfies:*

- (1)  $(x \rightarrow 0') \rightarrow y \approx (x \rightarrow y') \rightarrow y$
- (2)  $(0 \rightarrow x') \rightarrow (y \rightarrow x) \approx y \rightarrow x$
- (3)  $(y \rightarrow x)' \approx (0 \rightarrow x) \rightarrow (y \rightarrow x)'$
- (4)  $[x \rightarrow (y \rightarrow x)]' \approx (x \rightarrow y) \rightarrow x$
- (5)  $(y \rightarrow x) \rightarrow y \approx (0 \rightarrow x) \rightarrow y$
- (6)  $0 \rightarrow x \approx 0 \rightarrow (0 \rightarrow x)$
- (7)  $0 \rightarrow [(0 \rightarrow x) \rightarrow (0 \rightarrow y)'] \approx 0 \rightarrow (x \rightarrow y)$
- (8)  $[x' \rightarrow (0 \rightarrow y)]' \approx (0 \rightarrow x) \rightarrow (0 \rightarrow y)'$
- (9)  $0 \rightarrow (0 \rightarrow x)' \approx 0 \rightarrow x'$
- (10)  $0 \rightarrow (x' \rightarrow y)' \approx x \rightarrow (0 \rightarrow y')$
- (11)  $[(x \rightarrow 0') \rightarrow y]' \approx (0 \rightarrow x) \rightarrow y'$

- (12)  $0 \rightarrow [(0 \rightarrow x) \rightarrow y'] \approx x \rightarrow (0 \rightarrow y')$
- (13)  $0 \rightarrow (x \rightarrow y) \approx x \rightarrow (0 \rightarrow y)$
- (14)  $(x \rightarrow y) \rightarrow y' \approx y \rightarrow (x \rightarrow y)'$
- (15)  $(x' \rightarrow y) \rightarrow [(0 \rightarrow z) \rightarrow x'] \approx (0 \rightarrow y) \rightarrow [(0 \rightarrow z) \rightarrow x']$
- (16)  $0 \rightarrow (x \rightarrow y')' \approx 0 \rightarrow (x' \rightarrow y)$
- (17)  $x \rightarrow (y \rightarrow x') \approx y \rightarrow x'$
- (18)  $[(0 \rightarrow x) \rightarrow y] \rightarrow x \approx y \rightarrow x$
- (19)  $[0 \rightarrow (x \rightarrow y)] \rightarrow x \approx (0 \rightarrow y) \rightarrow x$
- (20)  $(0 \rightarrow x) \rightarrow (0 \rightarrow y) \approx x \rightarrow (0 \rightarrow y)$
- (21)  $x \rightarrow y \approx x \rightarrow (x \rightarrow y)$
- (22)  $[\{x \rightarrow (0 \rightarrow y)\} \rightarrow z]' \approx z \rightarrow [(x \rightarrow y) \rightarrow z]'$
- (23)  $[0 \rightarrow (x \rightarrow y)] \rightarrow y' \approx y \rightarrow (x \rightarrow y)'$
- (24)  $x \rightarrow [(y \rightarrow z) \rightarrow x]' \approx (0 \rightarrow y) \rightarrow [x \rightarrow (z \rightarrow x)']$
- (25)  $0 \rightarrow [(0 \rightarrow x) \rightarrow y] \approx x \rightarrow (0 \rightarrow y)$
- (26)  $x \rightarrow (y \rightarrow x)' \approx (y \rightarrow 0') \rightarrow x'$
- (27)  $[(x' \rightarrow y) \rightarrow (z \rightarrow x)'] \rightarrow [(y \rightarrow z) \rightarrow x] \approx (y \rightarrow z) \rightarrow x$
- (28)  $[\{0 \rightarrow (x \rightarrow y)'\} \rightarrow (0 \rightarrow y')']' \approx 0 \rightarrow (x \rightarrow y)'$
- (29)  $[[0 \rightarrow \{(x \rightarrow y) \rightarrow z\}] \rightarrow \{0 \rightarrow (y \rightarrow z)\}']' \approx 0 \rightarrow \{(x \rightarrow y) \rightarrow z\}$
- (30)  $[x \rightarrow (0 \rightarrow y)']' \approx x' \rightarrow (y \rightarrow 0')'$
- (31)  $[(0 \rightarrow x) \rightarrow y]' \approx y \rightarrow (x \rightarrow y)'$
- (32)  $[x \rightarrow (y \rightarrow 0')']' \approx x' \rightarrow (0 \rightarrow y)'$
- (33)  $(x \rightarrow y)' \rightarrow (0 \rightarrow x)' \approx y' \rightarrow x'$
- (34)  $(0 \rightarrow x)' \rightarrow (0 \rightarrow y)' \approx 0 \rightarrow (x' \rightarrow y')$
- (35)  $[(x \rightarrow y)' \rightarrow \{y \rightarrow (x \rightarrow y)'\}']' \approx (x \rightarrow y)'$
- (36)  $[\{(0 \rightarrow x) \rightarrow y\} \rightarrow (x \rightarrow y)']' \approx (0 \rightarrow x) \rightarrow y$
- (37)  $[\{x \rightarrow (y \rightarrow x)'\} \rightarrow x]' \approx x \rightarrow (y \rightarrow x)'.$

### 3 Partial order in Implication Zroupoids

Let  $\mathbf{A} = \langle A; \rightarrow, 0 \rangle \in \mathbf{I}$ . We define the operations  $\wedge$  and  $\vee$  on  $\mathbf{A}$  by:

- $x \wedge y := (x \rightarrow y')'$ ,
- $x \vee y := (x' \wedge y')'$ .

Note that the above definition of  $\wedge$  is a simultaneous generalization of the  $\wedge$  operation of De Morgan algebras and that of **SL** (= semilattices with least element 0). It is, of course, well known that the meet operation induces a partial order on both **DM** and **SL**, which naturally leads us to the following definition of a binary relation  $\sqsubseteq$  on algebras in **I**.

**Definition 3.1** *Let  $\mathbf{A} \in \mathbf{I}$ . We define the relation  $\sqsubseteq$  on  $A$  as follows:*

$$x \sqsubseteq y \text{ if and only if } x \wedge y = x \quad (\text{equivalently, } (x \rightarrow y')' = x).$$

For  $a, b \in A$ , we write

- $a \sqsubset b$  if  $a \sqsubseteq b$  and  $a \neq b$ ,
- $a \sqsupseteq b$  if  $b \sqsubseteq a$ , and
- $a \supset b$  if  $a \sqsupseteq b$  and  $a \neq b$ .

We already know from [3] that  $\langle A; \wedge, \vee \rangle$  is a lattice if and only if  $\mathbf{A}$  is a De Morgan Algebra, implying that  $\sqsubseteq$  is a partial order on  $A$ . We know (see [3]) that  $\sqsubseteq$  is also a partial order on algebras in **SL**. This fact led us naturally to consider the possibility of the existence of a subvariety **V** of **I**, containing both **SL** and **DM**, such that, for every algebra  $\mathbf{A}$  in **V**, the relation  $\sqsubseteq$  on  $\mathbf{A}$  is actually a partial order.

In this section we will prove our first main result which says that the subvariety **I**<sub>2,0</sub>, is a maximal subvariety of **I** with respect to the property that the relation  $\sqsubseteq$  is a partial order on every member of that variety. To achieve this end, we need to, first, prove that  $\sqsubseteq$  is indeed a partial order on every member of **I**<sub>2,0</sub>, which will be done using the following lemmas.

**Lemma 3.2** *Let  $\mathbf{A} \in \mathbf{I}_{2,0}$ . Then the relation  $\sqsubseteq$  is antisymmetric on  $\mathbf{A}$ .*

**Proof** Let  $a, b \in A$  such that  $a \sqsubseteq b$  and  $b \sqsubseteq a$ . Let  $c \in A$  be arbitrary. Then, using (I) and the hypothesis, one observes that  $(c \rightarrow a) \rightarrow b' = [(b \rightarrow c) \rightarrow (a \rightarrow b')']' = [(b \rightarrow c) \rightarrow a']'$ . Consequently,

$$(3.1) \quad (c \rightarrow a) \rightarrow b' = [(b \rightarrow c) \rightarrow a']', \text{ where } c \in A.$$

Hence,

$$\begin{aligned} a' &= (a \wedge b)' && \text{by hypothesis} \\ &= (a \rightarrow b')'' && \text{by definition of } \wedge \\ &= a \rightarrow b' \\ &= (a' \rightarrow a) \rightarrow b' && \text{using Lemma 2.4(d)} \\ &= [(b \rightarrow a') \rightarrow a]' && \text{from (3.1) with } c = a' \\ &= [(b \rightarrow a')'' \rightarrow a]' \\ &= (b' \rightarrow a)' && \text{by hypothesis,} \end{aligned}$$

and, therefore,

$$(3.2) \quad a' = (b' \rightarrow a)'$$

Now,

$$\begin{aligned} b' &= [b \rightarrow a']'' && \text{by hypothesis} \\ &= b \rightarrow a' \\ &= (0 \rightarrow a'') \rightarrow (b \rightarrow a') && \text{by Lemma 2.6 (2) with } x = a', y = b \\ &= (0 \rightarrow a) \rightarrow (b \rightarrow a')'' \\ &= (0 \rightarrow a) \rightarrow b' && \text{by hypothesis.} \end{aligned}$$

Thus,

$$(3.3) \quad b' = (0 \rightarrow a) \rightarrow b'.$$

Therefore,

$$\begin{aligned} a' &= [b' \rightarrow a]' && \text{from (3.2)} \\ &= [(b \rightarrow 0) \rightarrow a]' \\ &= (0 \rightarrow a) \rightarrow b' && \text{from (3.1) with } c = 0 \\ &= b' && \text{by (3.3).} \end{aligned}$$

Consequently, we have that  $a = a'' = b'' = b$ , thus proving that  $\sqsubseteq$  is antisymmetric on  $\mathbf{A}$ .  $\square$

Now, we turn to proving the transitivity of the relation  $\sqsubseteq$ . For this, we need the following lemmas. The proof of the following (technical) lemma is given in the Appendix.

**Lemma 3.3** *Let  $\mathbf{A} \in \mathbf{I}_{2,0}$  with  $a, b \in A$  such that  $a \sqsubseteq b$ . Let  $d \in A$  be arbitrary. Then*

- (1)  $(0 \rightarrow a') \rightarrow b = a' \rightarrow b$
- (2)  $b \rightarrow a' = (0 \rightarrow b) \rightarrow a'$
- (3)  $b \rightarrow a' = a'$
- (4)  $0 \rightarrow (a' \rightarrow b) = 0 \rightarrow a$
- (5)  $[(b \rightarrow d) \rightarrow a]' = (d \rightarrow a) \rightarrow b'$
- (6)  $(0 \rightarrow d) \rightarrow a' = [\{d \rightarrow (0 \rightarrow b')\} \rightarrow a]'$
- (7)  $a \rightarrow [(a' \rightarrow d) \rightarrow \{(0 \rightarrow a) \rightarrow b'\}] = (0 \rightarrow d) \rightarrow a'$
- (8)  $a \rightarrow [(d \rightarrow a) \rightarrow b'] = a \rightarrow (d \rightarrow a)'$
- (9)  $[0 \rightarrow (b \rightarrow d)] \rightarrow a = (0 \rightarrow d) \rightarrow a$
- (10)  $[b \rightarrow (a \rightarrow d)] \rightarrow a = (0 \rightarrow d) \rightarrow a$
- (11)  $b \rightarrow (0 \rightarrow a') = 0 \rightarrow a'$
- (12)  $[(d \rightarrow a) \rightarrow b']' = (b \rightarrow d) \rightarrow a$
- (13)  $a' \rightarrow b = b' \rightarrow a$
- (14)  $(d \rightarrow a') \rightarrow b = (d \rightarrow 0') \rightarrow (a' \rightarrow b)$

$$(15) \quad [(0 \rightarrow a') \rightarrow b]' = (0 \rightarrow a) \rightarrow b'$$

$$(16) \quad (a' \rightarrow b)' = (0 \rightarrow a) \rightarrow b'$$

$$(17) \quad b' \rightarrow [(b \rightarrow d) \rightarrow a] \sqsubseteq 0 \rightarrow b.$$

**Lemma 3.4** *Let  $\mathbf{A} \in \mathbf{I}_{2,0}$  and let  $a, b, e \in A$  such that  $(a \rightarrow b')' = a$  and  $(0 \rightarrow e') \rightarrow b = b$ , and let  $d \in A$  be arbitrary. Then*

$$(a) \quad b \rightarrow d = (0 \rightarrow (d \rightarrow e)) \rightarrow (b \rightarrow d)$$

$$(b) \quad (0 \rightarrow e) \rightarrow a' = a'$$

$$(c) \quad (0 \rightarrow e') \rightarrow a = a.$$

$$(d) \quad (0 \rightarrow e) \rightarrow [a \rightarrow (a \rightarrow d)] = a \rightarrow d.$$

**Proof**

(a)

$$\begin{aligned}
 b \rightarrow d &= [(0 \rightarrow e') \rightarrow b] \rightarrow d && \text{by hypothesis} \\
 &= [\{d' \rightarrow (0 \rightarrow e')\} \rightarrow (b \rightarrow d)'] \rightarrow [\{(0 \rightarrow e') \rightarrow b\} \rightarrow d] && \text{by Lemma 2.6 (27)} \\
 &\quad \text{using } x = d, y = 0 \rightarrow e', z = b \\
 &= [\{d' \rightarrow (0 \rightarrow e')\} \rightarrow (b \rightarrow d)'] \rightarrow (b \rightarrow d) && \text{by hypothesis} \\
 &= [\{d' \rightarrow (0 \rightarrow e')\} \rightarrow 0'] \rightarrow (b \rightarrow d) && \text{by Lemma 2.6 (1)} \\
 &= [0 \rightarrow \{d' \rightarrow (0 \rightarrow e')\}'] \rightarrow (b \rightarrow d) && \text{by Lemma 2.5 (a)} \\
 &= [0 \rightarrow \{(0 \rightarrow d) \rightarrow (0 \rightarrow e')'\}] \rightarrow (b \rightarrow d) && \text{by Lemma 2.6 (8)} \\
 &= [0 \rightarrow (d \rightarrow e)] \rightarrow (b \rightarrow d) && \text{by Lemma 2.6 (7)}.
 \end{aligned}$$

(b) Using Lemma 3.3 (3) (twice), and (a) with  $d = a'$ , we obtain  $[0 \rightarrow (a' \rightarrow e)] \rightarrow a' = [0 \rightarrow (a' \rightarrow e)] \rightarrow (b \rightarrow a') = b \rightarrow a' = a'$ . Hence,

$$(3.4) \quad [0 \rightarrow (a' \rightarrow e)] \rightarrow a' = a'.$$

Then,

$$\begin{aligned}
 (0 \rightarrow e) \rightarrow a' &= [0 \rightarrow (a' \rightarrow e)] \rightarrow a' && \text{by Lemma 2.6 (19) using } x = a', y = e \\
 &= a' && \text{by (3.4)}.
 \end{aligned}$$

(c)

$$\begin{aligned}
 (0 \rightarrow e') \rightarrow a &= [0 \rightarrow (0 \rightarrow e)'] \rightarrow a && \text{by Lemma 2.6 (9)} \\
 &= [(0 \rightarrow e) \rightarrow 0'] \rightarrow a && \text{by Lemma 2.5 (a)} \\
 &= [(0 \rightarrow e) \rightarrow a'] \rightarrow a && \text{by Lemma 2.6 (1)} \\
 &= a' \rightarrow a && \text{by (b)} \\
 &= a && \text{by Lemma 2.4 (d)}.
 \end{aligned}$$

(d)

$$\begin{aligned}
a \rightarrow d &= [(0 \rightarrow e') \rightarrow a] \rightarrow d && \text{by item (c)} \\
&= [\{d' \rightarrow (0 \rightarrow e')\} \rightarrow (a \rightarrow d)'] \rightarrow [\{(0 \rightarrow e') \rightarrow a\} \rightarrow d] && \text{by Lemma 2.6 (27) with } x = d, y = 0 \rightarrow e', z = a \\
&= [\{d' \rightarrow (0 \rightarrow e')\} \rightarrow (a \rightarrow d)'] \rightarrow (a \rightarrow d) && \text{by item (c)} \\
&= [\{d' \rightarrow (0 \rightarrow e')\} \rightarrow 0'] \rightarrow (a \rightarrow d) && \text{by Lemma 2.6 (1)} \\
&= [0 \rightarrow \{d' \rightarrow (0 \rightarrow e')\}'] \rightarrow (a \rightarrow d) && \text{by Lemma 2.5 (a)} \\
&= [0 \rightarrow \{(0 \rightarrow d) \rightarrow (0 \rightarrow e')'\}] \rightarrow (a \rightarrow d) && \text{by Lemma 2.6 (8) with } x = d, y = e' \\
&= [0 \rightarrow (d \rightarrow e)] \rightarrow (a \rightarrow d) && \text{by Lemma 2.6 (7)}
\end{aligned}$$

Thus,

$$(3.5) \quad a \rightarrow d = [0 \rightarrow (d \rightarrow e)] \rightarrow (a \rightarrow d).$$

Now,

$$\begin{aligned}
(0 \rightarrow e) \rightarrow [a \rightarrow (a \rightarrow d)] &= [0 \rightarrow [\{a \rightarrow (a \rightarrow d)\} \rightarrow e]] \rightarrow [a \rightarrow (a \rightarrow d)] \\
&\quad \text{by Lemma 2.6 (19) with } x = a \rightarrow (a \rightarrow d), y = e \\
&= [0 \rightarrow [\{a \rightarrow (a \rightarrow d)\} \rightarrow e]] \rightarrow [a \rightarrow \{a \rightarrow (a \rightarrow d)\}] \\
&\quad \text{by Lemma 2.6 (21)} \\
&= a \rightarrow [a \rightarrow (a \rightarrow d)] \\
&\quad \text{by (3.5) replacing } d \text{ with } a \rightarrow (a \rightarrow d) \\
&= a \rightarrow d \\
&\quad \text{by Lemma 2.6 (21)}.
\end{aligned}$$

Thus, (d) is proved and the proof of the lemma is complete.  $\square$

Each of the next three lemmas prove a crucial step in the proof of transitivity of  $\sqsubseteq$ .

**Lemma 3.5** *Let  $\mathbf{A} \in \mathbf{I}_{2,0}$  and let  $a, b, c \in A$  such that  $a \sqsubseteq b$  and  $b \sqsubseteq c$ . Let  $d, e, f \in A$  be arbitrary. Then*

- (1)  $(0 \rightarrow c') \rightarrow b = b$
- (2)  $(0 \rightarrow c) \rightarrow [a \rightarrow (a \rightarrow d)] = a \rightarrow d$
- (3)  $(0 \rightarrow c) \rightarrow (a \rightarrow d) = a \rightarrow d$
- (4)  $[0 \rightarrow ((0 \rightarrow b) \rightarrow c')] \rightarrow b = b$
- (5)  $\{d' \rightarrow [0 \rightarrow ((0 \rightarrow b) \rightarrow c')]\} \rightarrow (b \rightarrow d)' = (b \rightarrow d)'$
- (6)  $(b \rightarrow d) \rightarrow [e \rightarrow (b \rightarrow d)]' = [e \rightarrow 0'] \rightarrow (b \rightarrow d)'$
- (7)  $[b \rightarrow (a \rightarrow c')] \rightarrow a = a$
- (8)  $(0 \rightarrow b) \rightarrow (a \rightarrow d) = a \rightarrow d$



- (9)  $0 \rightarrow [b \rightarrow (a \rightarrow d)] = 0 \rightarrow (a \rightarrow d)$
- (10)  $0 \rightarrow [\{b \rightarrow (a \rightarrow d)\} \rightarrow e] = 0 \rightarrow [(a \rightarrow d) \rightarrow e]$
- (11)  $[0 \rightarrow (d' \rightarrow c)] \rightarrow (0 \rightarrow b)' = (0 \rightarrow d) \rightarrow (0 \rightarrow b)'$
- (12)  $0 \rightarrow (a' \rightarrow c) \sqsubseteq 0 \rightarrow b$
- (13)  $(0 \rightarrow a) \rightarrow (0 \rightarrow b)' = (0 \rightarrow a)'$
- (14)  $0 \rightarrow (a' \rightarrow c) = 0 \rightarrow a$
- (15)  $(d \rightarrow e) \rightarrow [\{b \rightarrow (a \rightarrow f)\}' \rightarrow (0 \rightarrow a)'] = (d \rightarrow e) \rightarrow [(a' \rightarrow b) \rightarrow (f' \rightarrow a')].$

**Proof** By hypothesis, we have  $(a \rightarrow b')' = a$  and  $(b \rightarrow c')' = b$ .

(1)

$$\begin{aligned}
(0 \rightarrow c') \rightarrow b &= (c \rightarrow 0') \rightarrow b && \text{by Lemma 2.5 (a)} \\
&= [(b' \rightarrow c) \rightarrow (0' \rightarrow b)']' && \text{by (I)} \\
&= [(b' \rightarrow c) \rightarrow b']' && \text{by Lemma 2.4 (a)} \\
&= [(0 \rightarrow c) \rightarrow b']' && \text{by Lemma 2.6 (5)} \\
&= [(c' \rightarrow 0') \rightarrow b']' && \text{by Lemma 2.5 (a)} \\
&= [(b'' \rightarrow c') \rightarrow (0' \rightarrow b')']'' && \text{from (I)} \\
&= (b'' \rightarrow c') \rightarrow (0' \rightarrow b')' \\
&= (b \rightarrow c') \rightarrow (0' \rightarrow b')' \\
&= (b \rightarrow c') \rightarrow b'' && \text{by Lemma 2.4 (a)} \\
&= (b \rightarrow c') \rightarrow b \\
&= b' \rightarrow b && \text{by hypothesis} \\
&= b && \text{by Lemma 2.4 (d)}.
\end{aligned}$$

(2) This is immediate from (1) and Lemma 3.4 (d) with  $e = c$ .

(3) Using Lemma 2.6 (21) and (2) we have that  $(0 \rightarrow c) \rightarrow (a \rightarrow d) = (0 \rightarrow c) \rightarrow [a \rightarrow (a \rightarrow d)] = a \rightarrow d$ , implying (3).

(4)

$$\begin{aligned}
[0 \rightarrow ((0 \rightarrow b) \rightarrow c')] \rightarrow b &= \{(b' \rightarrow 0) \rightarrow [((0 \rightarrow b) \rightarrow c') \rightarrow b]'\}' && \text{by (I)} \\
&= \{b \rightarrow [((0 \rightarrow b) \rightarrow c') \rightarrow b]'\}' \\
&= \{b \rightarrow (c' \rightarrow b)'\}' && \text{by Lemma 2.6 (18)} \\
&= \{(b' \rightarrow b) \rightarrow (c' \rightarrow b)'\}' && \text{by Lemma 2.4 (d)} \\
&= (b \rightarrow c') \rightarrow b && \text{by (I)} \\
&= (b \rightarrow c')'' \rightarrow b \\
&= b' \rightarrow b && \text{by hypothesis} \\
&= b && \text{by Lemma 2.4 (d)}.
\end{aligned}$$

(5)

$$\begin{aligned}
\{d' \rightarrow [0 \rightarrow ((0 \rightarrow b) \rightarrow c')]\} \rightarrow (b \rightarrow d)' &= \{[[0 \rightarrow ((0 \rightarrow b) \rightarrow c')] \rightarrow b] \rightarrow d\}' && \text{by (I)} \\
&= (b \rightarrow d)' && \text{by (4)}.
\end{aligned}$$

(6)

$$\begin{aligned}
(b \rightarrow d) \rightarrow [e \rightarrow (b \rightarrow d)]' &= [e \rightarrow (b \rightarrow d)] \rightarrow (b \rightarrow d)' && \text{by Lemma 2.6 (14) with } x = e, y = b \rightarrow d \\
&= [e \rightarrow 0'] \rightarrow (b \rightarrow d)' && \text{by Lemma 2.6 (1).}
\end{aligned}$$

(7)

$$\begin{aligned}
[b \rightarrow (a \rightarrow c')] \rightarrow a &= [(a' \rightarrow b) \rightarrow \{(a \rightarrow c') \rightarrow a\}'']' && \text{by (I)} \\
&= [(a' \rightarrow b) \rightarrow \{(0 \rightarrow c') \rightarrow a\}'']' && \text{by Lemma 2.6 (5)} \\
&= [(a' \rightarrow b) \rightarrow a']' && \text{by (1) and Lemma 3.4 (c)} \\
&= [(0 \rightarrow b) \rightarrow a']' && \text{by Lemma 2.6 (5)} \\
&= (a \rightarrow 0) \rightarrow (b \rightarrow a')' && \text{by (I)} \\
&= a' \rightarrow (b \rightarrow a')' \\
&= a' \rightarrow a'' && \text{by Lemma 3.3 (3)} \\
&= a' \rightarrow a \\
&= a && \text{by Lemma 2.4 (d).}
\end{aligned}$$

(8)

$$\begin{aligned}
(0 \rightarrow b') \rightarrow b &= (0 \rightarrow 0') \rightarrow b && \text{by Lemma 2.6 (1)} \\
&= (0'' \rightarrow 0') \rightarrow b \\
&= 0' \rightarrow b && \text{by Lemma 2.4 (d)} \\
&= b && \text{by Lemma 2.4 (a).}
\end{aligned}$$

Hence, by the hypothesis, together with Lemma 3.4 (d), we obtain that  $(0 \rightarrow b) \rightarrow \{a \rightarrow (a \rightarrow d)\} = a \rightarrow d$ . Hence, by Lemma 2.6 (21), we have  $(0 \rightarrow b) \rightarrow (a \rightarrow d) = a \rightarrow d$ .

(9)

$$\begin{aligned}
0 \rightarrow (a \rightarrow d) &= 0 \rightarrow [(0 \rightarrow b) \rightarrow (a \rightarrow d)] && \text{by (8)} \\
&= b \rightarrow [0 \rightarrow (a \rightarrow d)] && \text{by Lemma 2.6 (25) with } x = b, y = a \rightarrow d \\
&= 0 \rightarrow [b \rightarrow (a \rightarrow d)]. && \text{by Lemma 2.6 (13).}
\end{aligned}$$

(10)

$$\begin{aligned}
0 \rightarrow [\{b \rightarrow (a \rightarrow d)\} \rightarrow e] &= [b \rightarrow (a \rightarrow d)] \rightarrow (0 \rightarrow e) && \text{by Lemma 2.6 (13)} \\
&= 0 \rightarrow [[0 \rightarrow \{b \rightarrow (a \rightarrow d)\}] \rightarrow e] && \text{by Lemma 2.6 (25)} \\
&= 0 \rightarrow [\{0 \rightarrow (a \rightarrow d)\} \rightarrow e] && \text{by (9)} \\
&= (a \rightarrow d) \rightarrow (0 \rightarrow e) && \text{by Lemma 2.6 (25)} \\
&= 0 \rightarrow [(a \rightarrow d) \rightarrow e] && \text{by Lemma 2.6 (13).}
\end{aligned}$$

(11)

$$\begin{aligned}
[0 \rightarrow (d' \rightarrow c)] \rightarrow (0 \rightarrow b)' &= [0 \rightarrow (d' \rightarrow c)] \rightarrow (b' \rightarrow 0')' && \text{by Lemma 2.5 (a)} \\
&= [\{(d' \rightarrow c) \rightarrow b'\} \rightarrow 0']' && \text{by (I)} \\
&= [\{(b \rightarrow d') \rightarrow (c \rightarrow b')'\} \rightarrow 0']' && \text{by (I)} \\
&= [\{(b \rightarrow d') \rightarrow b''\}' \rightarrow 0']' && \text{by Lemma 3.3 (3)} \\
&= [\{(b \rightarrow d') \rightarrow b\}' \rightarrow 0']' \\
&= [\{(0 \rightarrow d') \rightarrow b\}' \rightarrow 0']' && \text{by Lemma 2.6 (5)} \\
&= [0 \rightarrow \{(0 \rightarrow d') \rightarrow b\}]' && \text{by Lemma 2.5 (a)} \\
&= [(0 \rightarrow d') \rightarrow (0 \rightarrow b)]' && \text{by Lemma 2.6 (13)} \\
&= [(d \rightarrow 0') \rightarrow (0 \rightarrow b)]' \\
&= (0 \rightarrow d) \rightarrow (0 \rightarrow b)' && \text{by Lemma 2.6 (11).}
\end{aligned}$$

$$\begin{aligned}
(12) \quad 0 \rightarrow (a' \rightarrow c) &= 0 \rightarrow [(a \rightarrow b')'' \rightarrow c] && \text{by hyphotesis} \\
&= 0 \rightarrow [(a \rightarrow b') \rightarrow c] \\
&\sqsubseteq 0 \rightarrow (b' \rightarrow c) && \text{by Lemma 2.6 (29)} \\
&= 0 \rightarrow b && \text{by hyphotesis and Lemma 3.3 (4).}
\end{aligned}$$

$$\begin{aligned}
(13) \quad (0 \rightarrow a) \rightarrow (0 \rightarrow b)' &= [a' \rightarrow (0 \rightarrow b)]' && \text{by Lemma 2.6 (8)} \\
&= [0 \rightarrow (a' \rightarrow b)]' && \text{by Lemma 2.6 (13)} \\
&= (0 \rightarrow a)'. && \text{by hyphotesis and Lemma 3.3 (4).}
\end{aligned}$$

$$\begin{aligned}
(14) \quad 0 \rightarrow (a' \rightarrow c) &= [\{0 \rightarrow (a' \rightarrow c)\} \rightarrow (0 \rightarrow b)']' && \text{by (12)} \\
&= [(0 \rightarrow a) \rightarrow (0 \rightarrow b)']' && \text{by (11) with } d = a \\
&= (0 \rightarrow a)'' && \text{by (13)} \\
&= 0 \rightarrow a.
\end{aligned}$$

$$\begin{aligned}
(15) \quad (d \rightarrow e) \rightarrow [(a' \rightarrow b) \rightarrow (f' \rightarrow a')] &= (d \rightarrow e) \rightarrow [\{(0 \rightarrow a') \rightarrow b\} \rightarrow (f' \rightarrow a')] \\
&\quad \text{by Lemma 3.3 (1)} \\
&= (d \rightarrow e) \rightarrow [\{(0 \rightarrow a') \rightarrow b\} \rightarrow \{(f \rightarrow 0) \rightarrow a'\}] \\
&= (d \rightarrow e) \rightarrow [\{(0 \rightarrow a') \rightarrow b\} \rightarrow \{(a \rightarrow f) \rightarrow (0 \rightarrow a')'\}] \\
&\quad \text{by (I)} \\
&= (d \rightarrow e) \rightarrow [\{b \rightarrow (a \rightarrow f)\} \rightarrow (0 \rightarrow a')'] \\
&\quad \text{by (I)} \\
&= (d \rightarrow e) \rightarrow [\{b \rightarrow (a \rightarrow f)\}' \rightarrow (a' \rightarrow 0)'] \\
&\quad \text{by (30) with } x = b \rightarrow (a \rightarrow f) \text{ and } y = a' \\
&= (d \rightarrow e) \rightarrow [\{b \rightarrow (a \rightarrow f)\}' \rightarrow (0 \rightarrow a)'] \\
&\quad \text{by Lemma 2.5 (a).}
\end{aligned}$$

Hence, we have  $(d \rightarrow e) \rightarrow [\{b \rightarrow (a \rightarrow f)\}' \rightarrow (0 \rightarrow a)'] = (d \rightarrow e) \rightarrow [(a' \rightarrow b) \rightarrow (f' \rightarrow a')]$ .

□

**Lemma 3.6** *Let  $\mathbf{A} \in \mathbf{I}_{2,0}$  and let  $a, b, c \in A$  such that  $a \sqsubseteq b$  and  $b \sqsubseteq c$ . Let  $d \in A$  be arbitrary. Then*

- (a)  $[c \rightarrow (b \rightarrow a')] \rightarrow b = (0 \rightarrow a') \rightarrow b$
- (b)  $(c \rightarrow a') \rightarrow b = a' \rightarrow b$
- (c)  $(a' \rightarrow b) \rightarrow (c \rightarrow a') = c \rightarrow a'$
- (d)  $c \rightarrow a' = a \rightarrow [b \rightarrow (a \rightarrow c')]$
- (e)  $0 \rightarrow (a \rightarrow d) = 0 \rightarrow [c \rightarrow (a \rightarrow d)]$

$$(f) \ (d \rightarrow a) \rightarrow d \sqsubseteq (a' \rightarrow b) \rightarrow d$$

$$(g) \ (a' \rightarrow b) \rightarrow c' = (0 \rightarrow a) \rightarrow b'$$

$$(h) \ 0 \rightarrow (a \rightarrow c') \sqsubseteq 0 \rightarrow a'$$

$$(i) \ 0 \rightarrow (a \rightarrow c') = 0 \rightarrow a'.$$

$$(j) \ c \rightarrow (a \rightarrow c') \sqsubseteq 0 \rightarrow (a \rightarrow c')$$

$$(k) \ c \rightarrow (a \rightarrow c') \sqsubseteq 0 \rightarrow a'$$

$$(l) \ (c \rightarrow (a \rightarrow c'))' \rightarrow (0 \rightarrow a)' = c \rightarrow (a \rightarrow c')$$

$$(m) \ a \rightarrow [b \rightarrow (a \rightarrow c')] = a \rightarrow c'$$

$$(n) \ c \rightarrow a' = a \rightarrow c'.$$

### Proof

$$(a) \text{ Since } (b \rightarrow c')' = b, \text{ by Lemma 3.3 (10) with } d = a', \text{ we have } (c \rightarrow (b \rightarrow a')) \rightarrow b = (0 \rightarrow a') \rightarrow b.$$

(b)

$$\begin{aligned} (c \rightarrow a') \rightarrow b &= [c \rightarrow (b \rightarrow a')] \rightarrow b \quad \text{by Lemma 3.3 (3)} \\ &= (0 \rightarrow a') \rightarrow b \quad \text{by (a),} \end{aligned}$$

from which we get  $(c \rightarrow a') \rightarrow b = (0 \rightarrow a') \rightarrow b$ , which, together with Lemma 3.3 (1), implies  $(c \rightarrow a') \rightarrow b = a' \rightarrow b$ .

(c)

$$\begin{aligned} c \rightarrow a' &= (0 \rightarrow a) \rightarrow (c \rightarrow a') && \text{by Lemma 2.6 (2) with } x = a', y = c \\ &= [0 \rightarrow (a' \rightarrow b)] \rightarrow (c \rightarrow a') && \text{by Lemma 3.3 (4)} \\ &= [0 \rightarrow \{(c \rightarrow a') \rightarrow b\}] \rightarrow (c \rightarrow a') && \text{by (b)} \\ &= [(c \rightarrow a') \rightarrow (0 \rightarrow b)] \rightarrow (c \rightarrow a') && \text{by Lemma 2.6 (13)} \\ &= [0 \rightarrow (0 \rightarrow b)] \rightarrow (c \rightarrow a') && \text{by Lemma 2.6 (5)} \\ &= (0 \rightarrow b) \rightarrow (c \rightarrow a') && \text{by Lemma 2.6 (6)} \\ &= [(c \rightarrow a') \rightarrow b] \rightarrow (c \rightarrow a') && \text{by Lemma 2.6 (5)} \\ &= (a' \rightarrow b) \rightarrow (c \rightarrow a') && \text{by (b).} \end{aligned}$$

(d)

$$\begin{aligned}
c \rightarrow a' &= (0 \rightarrow a) \rightarrow (c \rightarrow a') && \text{by Lemma 2.6 (2)} \\
&= (0 \rightarrow a) \rightarrow [(a' \rightarrow b) \rightarrow (c \rightarrow a')] && \text{by (c)} \\
&= (0 \rightarrow a) \rightarrow [(a' \rightarrow b) \rightarrow (c'' \rightarrow a')] \\
&= (0 \rightarrow a) \rightarrow [\{b \rightarrow (a \rightarrow c')\}' \rightarrow (0 \rightarrow a)'] && \text{by Lemma 3.5 (15) with } d=0, e=a, f=c' \\
&= (0 \rightarrow a) \rightarrow [\{b \rightarrow (a \rightarrow c')\}' \rightarrow \{0 \rightarrow (a' \rightarrow c)\}'] && \text{by Lemma 3.5 (14)} \\
&= (0 \rightarrow a) \rightarrow [\{b \rightarrow (a \rightarrow c')\}' \rightarrow \{0 \rightarrow (a \rightarrow c')'\}] && \text{by Lemma 2.6 (16)} \\
&= (0 \rightarrow a) \rightarrow [\{b \rightarrow (a \rightarrow c')\}' \rightarrow [0 \rightarrow \{b \rightarrow (a \rightarrow c')\}']] && \text{by Lemma 3.5(10) with } d=c', e=0 \\
&= [b \rightarrow (a \rightarrow c')]' \rightarrow [(a \rightarrow 0) \rightarrow \{b \rightarrow (a \rightarrow c')\}']' && \text{by Lemma 2.6 (24) with } x=[b \rightarrow (a \rightarrow c')]', y=a, z=0 \\
&= [\{0 \rightarrow (a \rightarrow 0)\} \rightarrow \{b \rightarrow (a \rightarrow c')\}']' && \text{by Lemma 2.6 (4) and (5) with } x=a \rightarrow 0, y=[b \rightarrow (a \rightarrow c')]' \\
&= [\{a \rightarrow (0 \rightarrow 0)\} \rightarrow \{b \rightarrow (a \rightarrow c')\}']' && \text{by Lemma 2.6 (13)} \\
&= [(a \rightarrow 0') \rightarrow \{b \rightarrow (a \rightarrow c')\}']' \\
&= [\{b \rightarrow (a \rightarrow c')\} \rightarrow [a \rightarrow \{b \rightarrow (a \rightarrow c')\}']]' && \text{by Lemma 3.5(6) with } e=a, d=a \rightarrow c' \\
&= [\{b \rightarrow (a \rightarrow c')\} \rightarrow a] \rightarrow [b \rightarrow (a \rightarrow c')] && \text{by Lemma 2.6 (4)} \\
&= a \rightarrow [b \rightarrow (a \rightarrow c')] && \text{by Lemma 3.5(7).}
\end{aligned}$$

(e)

$$\begin{aligned}
0 \rightarrow (a \rightarrow d) &= 0 \rightarrow [(0 \rightarrow c) \rightarrow (a \rightarrow d)] && \text{by Lemma 3.5(3)} \\
&= c \rightarrow [0 \rightarrow (a \rightarrow d)] && \text{by Lemma 2.6 (25)} \\
&= 0 \rightarrow [c \rightarrow (a \rightarrow d)] && \text{by Lemma 2.6 (13).}
\end{aligned}$$

(f)

$$\begin{aligned}
(d \rightarrow a) \rightarrow d &= (0 \rightarrow a) \rightarrow d && \text{by Lemma 2.6 (5)} \\
&= [0 \rightarrow (a' \rightarrow b)] \rightarrow d && \text{by Lemma 3.3 (4)} \\
&\sqsubseteq (a' \rightarrow b) \rightarrow d && \text{by Lemma 2.6 (36).}
\end{aligned}$$

(g)

$$\begin{aligned}
(a' \rightarrow b) \rightarrow c' &= [(c \rightarrow a') \rightarrow (b \rightarrow c')']' && \text{by (I)} \\
&= [(c \rightarrow a') \rightarrow b]' && \text{by hypothesis} \\
&= [\{c \rightarrow (b \rightarrow a')\} \rightarrow b]' && \text{by Lemma 3.3 (3)} \\
&= [(0 \rightarrow a') \rightarrow b]' && \text{by Lemma 3.3 (10) with } d=a' \text{ since } b \sqsubseteq c \\
&= [(b \rightarrow a') \rightarrow b]' && \text{by Lemma 2.6 (5)} \\
&= [(b \rightarrow a') \rightarrow b'']' \\
&= [(b \rightarrow a') \rightarrow (0' \rightarrow b')']' && \text{by Lemma 2.4 (a)} \\
&= (a' \rightarrow 0') \rightarrow b' && \text{by (I)} \\
&= (0 \rightarrow a) \rightarrow b'. && \text{by Lemma 2.5 (a).}
\end{aligned}$$

Hence, one has  $(a' \rightarrow b) \rightarrow c' = (0 \rightarrow a) \rightarrow b'$ .

(h) From Lemma 3.5 (1), we have  $(0 \rightarrow c') \rightarrow b = b$ . Hence, we can use Lemma 3.3. Therefore, we have

$$\begin{aligned}
0 \rightarrow (a \rightarrow c') &= 0 \rightarrow [\{(0 \rightarrow c') \rightarrow a\} \rightarrow c'] && \text{by Lemma 3.4 (c) and Lemma 3.5(1)} \\
&= [(0 \rightarrow c') \rightarrow a] \rightarrow (0 \rightarrow c') && \text{by Lemma 2.6 (13)} \\
&\sqsubseteq (a' \rightarrow b) \rightarrow (0 \rightarrow c') && \text{by (f) with } d = 0 \rightarrow c' \\
&= 0 \rightarrow [(a' \rightarrow b) \rightarrow c'] && \text{by Lemma 2.6 (13)} \\
&= 0 \rightarrow [(0 \rightarrow a) \rightarrow b'] && \text{by (g)} \\
&= 0 \rightarrow [(b \rightarrow 0) \rightarrow (a \rightarrow b')]' && \text{by (I)} \\
&= 0 \rightarrow [b' \rightarrow (a \rightarrow b')]' && \\
&= 0 \rightarrow (b' \rightarrow a)' && \text{by hypothesis} \\
&= 0 \rightarrow (b \rightarrow a') && \text{by Lemma 2.6 (16)} \\
&= 0 \rightarrow a' && \text{by Lemma 3.3 (3).}
\end{aligned}$$

(i)

$$\begin{aligned}
0 \rightarrow a' &= 0 \rightarrow (a \rightarrow 0) \\
&= 0 \rightarrow [c \rightarrow (a \rightarrow 0)] && \text{by (e)} \\
&= 0 \rightarrow (c \rightarrow a') \\
&= 0 \rightarrow [(a \rightarrow c')' \rightarrow (0 \rightarrow a)'] && \text{by Lemma 2.6 (33)} \\
&= [0 \rightarrow (a \rightarrow c')]' \rightarrow (0 \rightarrow a)' && \text{by Lemma 2.6 (34) and Lemma 2.6 (6)} \\
&= [0 \rightarrow (a \rightarrow c')]' \rightarrow (a' \rightarrow 0')' && \text{by Lemma 2.5 (a)} \\
&= [\{0 \rightarrow (a \rightarrow c')\} \rightarrow (0 \rightarrow a')]' && \text{by Lemma 2.5 (30) with } x = 0 \rightarrow (a \rightarrow c'), y = a' \\
&= 0 \rightarrow (a \rightarrow c') && \text{by (h).}
\end{aligned}$$

(j)

$$\begin{aligned}
[\{c \rightarrow (a \rightarrow c')\} \rightarrow \{0 \rightarrow (a \rightarrow c')\}]' &= [\{0 \rightarrow (a \rightarrow c')\} \rightarrow c] \rightarrow [(a \rightarrow c') \rightarrow \{0 \rightarrow (a \rightarrow c')\}]' \\
&&& \text{by (I)} \\
&= [\{0 \rightarrow (a \rightarrow c')\} \rightarrow c] \rightarrow [\{(a \rightarrow c') \rightarrow 0\} \rightarrow (a \rightarrow c')] \\
&&& \text{by Lemma 2.6 (4)} \\
&= [\{0 \rightarrow (a \rightarrow c')\} \rightarrow c] \rightarrow [(a \rightarrow c')' \rightarrow (a \rightarrow c')] \\
&= [\{0 \rightarrow (a \rightarrow c')\} \rightarrow c] \rightarrow (a \rightarrow c') \\
&&& \text{by Lemma 2.4 (d)} \\
&= c \rightarrow (a \rightarrow c') \\
&&& \text{by Lemma 2.6 (18) with } x = a \rightarrow c', y = c.
\end{aligned}$$

(k) From (j) we have that  $c \rightarrow (a \rightarrow c') \sqsubseteq 0 \rightarrow (a \rightarrow c')$ . Then using (i) we get  $c \rightarrow (a \rightarrow c') \sqsubseteq 0 \rightarrow a'$ .

(l)

$$\begin{aligned}
[c \rightarrow (a \rightarrow c')]' \rightarrow (0 \rightarrow a)' &= [c \rightarrow (a \rightarrow c')]' \rightarrow (a' \rightarrow 0')' && \text{by Lemma 2.5 (a)} \\
&= [\{c \rightarrow (a \rightarrow c')\} \rightarrow (0 \rightarrow a')]' && \text{by Lemma 2.6 (30)} \\
&= c \rightarrow (a \rightarrow c') && \text{by (k).}
\end{aligned}$$

(m)

$$\begin{aligned}
a \rightarrow [b \rightarrow (a \rightarrow c')] &= c \rightarrow a' && \text{by (d)} \\
&= c'' \rightarrow a' \\
&= (a \rightarrow c')' \rightarrow (0 \rightarrow a)' && \text{by Lemma 2.6 (33)} \\
&= [c \rightarrow (a \rightarrow c')]' \rightarrow (0 \rightarrow a)' && \text{by Lemma 2.6 (17)} \\
&= c \rightarrow (a \rightarrow c') && \text{by (l)} \\
&= a \rightarrow c' && \text{by Lemma 2.6 (17).}
\end{aligned}$$

(n) From (d) and (m), we get  $c \rightarrow a' = a \rightarrow c'$ .

□

**Lemma 3.7** *Let  $\mathbf{A} \in \mathbf{I}_{2,0}$  and let  $a, b, c \in A$  such that  $a \sqsubseteq b$  and  $b \sqsubseteq c$ . Then*

- (a)  $c' \rightarrow [(c \rightarrow d) \rightarrow b] \sqsubseteq c$
- (b)  $0 \rightarrow a' = c \rightarrow (0 \rightarrow a')$
- (c)  $c' \rightarrow (a' \rightarrow b) \sqsubseteq c$
- (d)  $(0 \rightarrow a') \rightarrow b = (c \rightarrow a') \rightarrow b$
- (e)  $c' \rightarrow (a' \rightarrow b) \sqsubseteq 0 \rightarrow c$
- (f)  $[(0 \rightarrow a) \rightarrow b'] \rightarrow c = c' \rightarrow (a' \rightarrow b)$
- (g)  $a' \rightarrow c = c' \rightarrow (a' \rightarrow b)$
- (h)  $a' \rightarrow c \sqsubseteq c$
- (i)  $a' \rightarrow c = (0 \rightarrow a') \rightarrow c$ .

**Proof**

- (a)
 
$$\begin{aligned}
 c' \rightarrow [(c \rightarrow d) \rightarrow b] &= c' \rightarrow [(c \rightarrow d) \rightarrow (b \rightarrow c')'] && \text{by hypothesis} \\
 &= c' \rightarrow [(d \rightarrow b) \rightarrow c']' && \text{by (I)} \\
 &\sqsubseteq c'' && \text{by Lemma 2.6 (37)} \\
 &= c.
 \end{aligned}$$
- (b)
 
$$\begin{aligned}
 0 \rightarrow a' &= b \rightarrow (0 \rightarrow a') && \text{by Lemma 3.3 (11)} \\
 &= [0 \rightarrow \{(0 \rightarrow a') \rightarrow c\}] \rightarrow [b \rightarrow (0 \rightarrow a')] && \text{by Lemma 3.5 (1)} \\
 & && \text{and Lemma 3.4 (a)} \\
 & && \text{with } d = 0 \rightarrow a', e = c \\
 &= [0 \rightarrow \{(0 \rightarrow a') \rightarrow c\}] \rightarrow (0 \rightarrow a') && \text{by Lemma 3.3 (11)} \\
 &= [(0 \rightarrow a') \rightarrow (0 \rightarrow c)] \rightarrow (0 \rightarrow a') && \text{by Lemma 2.6 (13)} \\
 &= [0 \rightarrow (0 \rightarrow c)] \rightarrow (0 \rightarrow a') && \text{by Lemma 2.6 (5)} \\
 &= (0 \rightarrow c) \rightarrow (0 \rightarrow a') && \text{by Lemma 2.6 (21)} \\
 &= c \rightarrow (0 \rightarrow a') && \text{by Lemma 2.6 (20)}.
 \end{aligned}$$
- (c)
 
$$\begin{aligned}
 c' \rightarrow (a' \rightarrow b) &= c' \rightarrow [(0 \rightarrow a') \rightarrow b] && \text{by Lemma 3.3 (1)} \\
 &= c' \rightarrow [\{c \rightarrow (0 \rightarrow a')\} \rightarrow b] && \text{by (b)} \\
 &\sqsubseteq c && \text{by (a) with } d = 0 \rightarrow a'.
 \end{aligned}$$

$$\begin{aligned}
(d) \quad (0 \rightarrow a') \rightarrow b &= [c \rightarrow (0 \rightarrow a')] \rightarrow b && \text{by (b)} \\
&= [(b' \rightarrow c) \rightarrow \{(0 \rightarrow a') \rightarrow b\}']' && \text{by (I)} \\
&= [(b' \rightarrow c) \rightarrow \{(b' \rightarrow 0) \rightarrow (a' \rightarrow b)'\}]' && \text{by (I) and } x'' \approx x \\
&= [(b' \rightarrow c) \rightarrow \{b \rightarrow (a' \rightarrow b)'\}]' && \\
&= [(b' \rightarrow c) \rightarrow \{(a' \rightarrow 0') \rightarrow b'\}]' && \text{by Lemma 2.6 (26)} \\
&= [(b' \rightarrow c) \rightarrow \{(0 \rightarrow a) \rightarrow b'\}]' && \text{by Lemma 2.5 (a)} \\
&= [(b' \rightarrow c) \rightarrow (a' \rightarrow b)']' && \text{by Lemma 3.3 (16)} \\
&= (c \rightarrow a') \rightarrow b && \text{by (I).}
\end{aligned}$$

$$\begin{aligned}
(e) \quad c' \rightarrow (a' \rightarrow b) &= c' \rightarrow [(0 \rightarrow a') \rightarrow b] && \text{by Lemma 3.3 (15) and Lemma 3.3 (16)} \\
&= c' \rightarrow [(c \rightarrow a') \rightarrow b] && \text{by (d)} \\
&\sqsubseteq 0 \rightarrow c. && \text{by Lemma 3.3 (17) with } d = a'.
\end{aligned}$$

$$\begin{aligned}
(f) \quad c' \rightarrow (a' \rightarrow b) &= [\{c' \rightarrow (a' \rightarrow b)\} \rightarrow (0 \rightarrow c)']' && \text{by (e)} \\
&= [(a' \rightarrow b) \rightarrow 0] \rightarrow c && \text{by (I)} \\
&= (a' \rightarrow b)' \rightarrow c && \\
&= [(0 \rightarrow a) \rightarrow b'] \rightarrow c && \text{by Lemma 3.3 (16).}
\end{aligned}$$

$$\begin{aligned}
(g) \quad c' \rightarrow (a' \rightarrow b) &= ((0 \rightarrow a) \rightarrow b') \rightarrow c && \text{by (f)} \\
&= [(0 \rightarrow a) \rightarrow 0'] \rightarrow (b' \rightarrow c) && \text{by Lemma 3.3 (14) with } d = 0 \rightarrow a \\
&= [(a' \rightarrow 0') \rightarrow 0'] \rightarrow (b' \rightarrow c) && \text{by Lemma 2.5 (a)} \\
&= [(a' \rightarrow 0) \rightarrow 0'] \rightarrow (b' \rightarrow c) && \text{by Lemma 2.6 (1)} \\
&= [a'' \rightarrow 0'] \rightarrow (b' \rightarrow c) && \\
&= (a \rightarrow 0') \rightarrow (b' \rightarrow c) && \\
&= (a \rightarrow b') \rightarrow c && \text{by Lemma 3.3 (14) with } d = a \\
&= a' \rightarrow c && \text{by hypothesis.}
\end{aligned}$$

(h) This is immediate from (g) and (c).

$$\begin{aligned}
(i) \quad (0 \rightarrow a') \rightarrow c &= (c \rightarrow a') \rightarrow c && \text{by Lemma 2.6 (5)} \\
&= [c \rightarrow (a' \rightarrow c)]' && \text{by Lemma 2.6 (4)} \\
&= [(a' \rightarrow c) \rightarrow c']' && \text{by Lemma 2.6 (14)} \\
&= a' \rightarrow c && \text{by (h).}
\end{aligned}$$

□

We are now ready to complete the proof of transitivity of  $\sqsubseteq$ .

**Theorem 3.8**  $\sqsubseteq$  is transitive.



**Proof** Let  $a, b, c \in A$  such that  $a \sqsubseteq b$  and  $b \sqsubseteq c$ . Observe that

$$\begin{aligned}
a' &= a \rightarrow 0 \\
&= (0 \rightarrow c) \rightarrow (a \rightarrow 0) && \text{by Lemma 3.5 (3) with } d = 0 \\
&= (0 \rightarrow c) \rightarrow a' \\
&= (a' \rightarrow c) \rightarrow a' && \text{by Lemma 2.6 (5)} \\
&= ((0 \rightarrow a') \rightarrow c) \rightarrow a' && \text{by Lemma 3.7 (i)} \\
&= c \rightarrow a' && \text{by Lemma 2.6 (18)} \\
&= a \rightarrow c' && \text{by Lemma 3.6 (n).}
\end{aligned}$$

Consequently,

$$a = a'' = (a \rightarrow c')',$$

implying  $a \sqsubseteq c$ . Hence,  $\sqsubseteq$  is transitive on  $\mathbf{A}$ .  $\square$

We are now prepared to present our first main theorem.

**Theorem 3.9** *The variety  $\mathbf{I}_{2,0}$  is a maximal subvariety of  $\mathbf{I}$  with respect to the property that the relation  $\sqsubseteq$  introduced in Definition 3.1 is a partial order.*

**Proof** Let  $\mathbf{A} \in \mathbf{I}_{2,0}$ . The relation  $\sqsubseteq$  is a partial order on  $A$  in view of Lemma 2.4 (c), Lemma 3.2, and Theorem 3.8.

Next, let  $\mathbf{V}$  be a subvariety of  $\mathbf{I}$  such that  $\sqsubseteq$  is a partial order on every algebra in  $\mathbf{V}$ . Now let  $\mathbf{A} \in \mathbf{V}$ . Reflexivity of  $\sqsubseteq$  implies that  $\mathbf{A} \models (x \rightarrow x')' \approx x$ . Therefore, by Lemma 2.4, we conclude that  $\mathbf{A} \in \mathbf{I}_{2,0}$ , and hence,  $\mathbf{V} \subseteq \mathbf{I}_{2,0}$ , completing the proof.  $\square$

## 4 A method to construct finite $\mathbf{I}_{2,0}$ -chains

Now that we know the relation  $\sqsubseteq$  is a partial order on algebras in  $\mathbf{I}_{2,0}$ , it is natural to consider those algebras in  $\mathbf{I}_{2,0}$ , in which  $\sqsubseteq$  is a total order.

**Definition 4.1** *Let  $\mathbf{A} \in \mathbf{I}$ . We say that  $\mathbf{A}$  is an  $\mathbf{I}_{2,0}$ -chain (chain, for short) if  $\mathbf{A} \in \mathbf{I}_{2,0}$  and the relation  $\sqsubseteq$  (see Definition 3.1) is totally ordered on  $A$ .*

In this section we describe a method of constructing finite  $\mathbf{I}_{2,0}$ -chains. But, first, we will present some examples of  $\mathbf{I}_{2,0}$ -chains that will foreshadow the method to construct finite  $\mathbf{I}_{2,0}$ -chains. We note that, in these examples, the number 0 is the constant element.

It is easy to see that the only 2-element  $\mathbf{I}_{2,0}$ -chains, up to isomorphism, are

$$\begin{array}{c|cc} \rightarrow: & 0 & 1 \\ \hline 0 & 1 & 1 \\ 1 & 0 & 1 \end{array} \text{ with } 0 \sqsubseteq 1.$$

$$\begin{array}{c|cc} \rightarrow: & -1 & 0 \\ \hline -1 & -1 & -1 \\ 0 & -1 & 0 \end{array} \text{ with } -1 \sqsubseteq 0$$

and the only 3-element  $\mathbf{I}_{2,0}$ -chains, up to isomorphism, are

$$\begin{array}{c|ccc} \rightarrow: & 0 & 1 & 2 \\ \hline 0 & 2 & 2 & 2 \\ 1 & 1 & 1 & 2 \\ 2 & 0 & 1 & 2 \end{array} \text{ with } 0 \sqsubseteq 1 \sqsubseteq 2,$$

$$\begin{array}{c|ccc} \rightarrow: & -1 & 0 & 1 \\ \hline -1 & -1 & -1 & -1 \\ 0 & -1 & 1 & 1 \\ 1 & -1 & 0 & 1 \end{array} \text{ with } -1 \sqsubseteq 0 \sqsubseteq 1,$$

$\rightarrow:$	-2	-1	0
-2	-2	-2	-2
-1	-2	-1	-1
0	-2	-1	0

with  $-2 \sqsubset -1 \sqsubset 0$ .

Note that, henceforth, we will use the symbol  $\leq$  to denote the natural order in  $\mathbb{Z}$ . Recall that  $\sqsubset$  is being used for the order (see Definition 3.1).

The next definition describes a general method to construct a class of finite  $\mathbf{I}_{2,0}$ -chains, generalizing the above examples. In the next section, we will show that, this method, in fact, yields, up to isomorphism, all finite  $\mathbf{I}_{2,0}$ -chains.

**Definition 4.2** Let  $k \in \mathbb{N}$ . Let  $m, n \in \omega$  be such that the interval  $[-n, m] \subseteq \mathbb{Z}$  with  $||[-n, m]| = k$  and  $0 \leq n, m \leq k - 1$ . The (auxiliary) functions  $p$  (predecessor) and  $*$  are defined on  $[-n, m]$  as follows:

$$p(x) = \begin{cases} x - 1 & \text{if } x > -n \\ -n & \text{if } x = -n, \end{cases}$$

and

$$x^* = \begin{cases} m & \text{if } x = 0 \\ x & \text{if } x < 0 \\ p((p(x))^*) & \text{if } x > 0. \end{cases}$$

For convenience, we write  $p(p(x)^*)$  for  $p((p(x))^*)$ . (Notice that the function  $*$  is defined recursively for  $x \geq 0$ .)

Define the algebra  $[-\mathbf{n}, \mathbf{m}]$  as follows:

$[-\mathbf{n}, \mathbf{m}] := \langle [-n, m]; \Rightarrow, 0 \rangle$ , where  $0 \in [-n, m]$  is the constant and  $\Rightarrow$  is defined by

$$x \Rightarrow y = \begin{cases} \max(x^*, y) & \text{if } x, y \geq 0 \\ \min(x, y) & \text{otherwise.} \end{cases}$$

We set  $x' := x \Rightarrow 0$ .

We shall now illustrate the method described in the above definition by applying it to construct a 6-element  $\mathbf{I}_{2,0}$ -chain.

Let  $k = 6$ , and consider the interval  $A = [-2, 3] = \{-2, -1, 0, 1, 2, 3\}$ . Since  $0 \Rightarrow 0 = \max(0^*, 0) = \max(3, 0) = 3$  and  $a \Rightarrow b = \min(a, b)$  if  $a < 0$  or  $b < 0$ , we arrive at the following partial table for  $\Rightarrow$ :

$\Rightarrow$	-2	-1	0	1	2	3
-2	-2	-2	-2	-2	-2	-2
-1	-2	-1	-1	-1	-1	-1
0	-2	-1	3	?	?	?
1	-2	-1	?	?	?	?
2	-2	-1	?	?	?	?
3	-2	-1	?	?	?	?

Next, we determine the operations  $p$  and  $*$ :

$x$	$x^*$
0	3
1	$p(p(1)^*) = p(0^*) = p(3) = 2$
2	$p(p(2)^*) = p(1^*) = p(2) = 1$
3	$p(p(3)^*) = p(2^*) = p(1) = 0$

$x$	$x \Rightarrow 0$
1	$\max(1^*, 0) = \max(2, 0) = 2$
2	$\max(2^*, 0) = \max(1, 0) = 1$
3	$\max(3^*, 0) = \max(0, 0) = 0$

Hence the table for  $\Rightarrow$  becomes:

$\Rightarrow$	-2	-1	0	1	2	3
-2	-2	-2	-2	-2	-2	-2
-1	-2	-1	-1	-1	-1	-1
0	-2	-1	3	?	?	?
1	-2	-1	2	?	?	?
2	-2	-1	1	?	?	?
3	-2	-1	0	?	?	?

Observe that  $0 \Rightarrow 1 = \max(0^*, 1) = \max(3, 1) = 3$ ,  $1 \Rightarrow 1 = \max(1^*, 1) = \max(2, 1) = 2$ ,  $2 \Rightarrow 1 = \max(2^*, 1) = \max(1, 1) = 1$  and  $3 \Rightarrow 1 = \max(3^*, 1) = \max(0, 1) = 1$ . Then we get

$\Rightarrow$	-2	-1	0	1	2	3
-2	-2	-2	-2	-2	-2	-2
-1	-2	-1	-1	-1	-1	-1
0	-2	-1	3	3	?	?
1	-2	-1	2	2	?	?
2	-2	-1	1	1	?	?
3	-2	-1	0	1	?	?

Iterating this process we obtain the following complete table for  $\Rightarrow$ :

$\Rightarrow$	-2	-1	0	1	2	3
-2	-2	-2	-2	-2	-2	-2
-1	-2	-1	-1	-1	-1	-1
0	-2	-1	3	3	3	3
1	-2	-1	2	2	2	3
2	-2	-1	1	1	2	3
3	-2	-1	0	1	2	3

Thus we have constructed the algebra  $[-\mathbf{n}, \mathbf{m}]$ . Observe that  $-2 \sqsubset -1 \sqsubset 0 \sqsubset 1 \sqsubset 2 \sqsubset 3$  and  $x'' = x^{**} = x$ . Also, it is routine to verify  $[-\mathbf{n}, \mathbf{m}] \in \mathbf{I}_{2,0}$ . Hence it is an  $\mathbf{I}_{2,0}$ -chain.

Returning to the general method, we now aim to prove that  $[-\mathbf{n}; \mathbf{m}]$  is an  $\mathbf{I}_{2,0}$ -chain. To prove this, we will need the following lemmas.

**Lemma 4.3** *If  $x \in [-\mathbf{n}, \mathbf{m}]$  and  $0 \leq x \leq m$  then  $x^* = m - x$  and, consequently,  $x^* \in [0, m]$ .*

**Proof** We prove this lemma by induction on the element  $x$ . Assume that  $x = 0$ . Then  $0^* = m = m - 0$ .

Next, suppose  $x > 0$ . Since  $-n \leq 0 < x$ , we have  $p(x) = x - 1$ . Hence, by inductive hypothesis, we have

$$(4.1) \quad p(x)^* = m - p(x) = m - (x - 1) = m - x + 1.$$

From  $x > 0$ , we can conclude that  $m - x + 1 \leq m$ . Also, since  $x \leq m$ , we obtain  $0 \leq m - x$ , thus  $-n - 1 < 0 \leq m - x$ , implying  $m - x + 1 > -n$ . Then we get  $p(m - x + 1) = m - x + 1 - 1$ . By (4.1),  $x^* = p((p(x))^*) = p(m - x + 1) = m - x$ , completing the induction. It is clear that  $x^* \in [0, m]$ .  $\square$

**Corollary 4.4** *If  $x \in [-n, m]$  then  $x' = x^*$ .*

**Proof** If  $x < 0$  we have that  $x' = x \Rightarrow 0 = \min(x, 0) = x = x^*$ . If  $x > 0$ , then by Lemma 4.3,  $x^* \geq 0$ , and hence  $x' = x \Rightarrow 0 = \max(x^*, 0) = x^*$ .  $\square$

**Lemma 4.5** *If  $x \in [-n, m]$  then  $x'' = x$ .*

**Proof** We consider the following cases:

- If  $x < 0$ , then  $x^* = x$ , and hence  $x^{**} = x$ .

- If  $x \geq 0$ ,

$$\begin{aligned} x^{**} &= (m - x)^* && \text{by Lemma 4.3 since } 0 < x \leq m \\ &= m - (m - x) && \text{by Lemma 4.3 since } 0 \leq m - x \leq m \\ &= x. \end{aligned}$$

Consequently, by Corollary 4.4,  $x'' = x$ .  $\square$

**Lemma 4.6** *If  $x, y \in [-n, m]$  and  $0 \leq x \leq y$  then  $x^* \geq y^*$ .*

**Proof** We prove this lemma by induction on the element  $x$ . If  $x = 0$ ,  $x^* = 0^* = m \geq y^*$  by Lemma 4.3.

Now assume that  $x > 0$ . Since  $0 < x \leq y$ , we have that  $x^* = p(p(x)^*)$  and  $y^* = p(p(y)^*)$ . Note that  $0 \leq p(x) \leq p(y)$ . Then, by induction hypothesis, we get  $p(y)^* \leq p(x)^*$ . Hence  $x^* = p(p(x)^*) \geq p(p(y)^*) = y^*$ .  $\square$

**Lemma 4.7** *Let  $k \in \mathbb{N}$ . Let  $m, n \in \omega$  be such that the interval  $[-n, m] \subseteq \mathbb{Z}$  with  $||[-n, m]| = k$  and  $0 \leq n, m \leq k - 1$ . Then,  $[-n, m] \in \mathbf{I}_{2,0}$ .*

**Proof** The proof that  $\langle [-n, m]; \Rightarrow, 0 \rangle$  satisfies the identity (I) is long and computational, but routine. Hence we leave the verification to the reader with the recommendation that the following cases be considered, where  $i, j, k \in [-n, m]$ :

- |  |                                      |
|--|--------------------------------------|
| (1) $i, j, k \geq 0, i^* \geq j, i \geq k$       | (7) $i \geq 0, j < 0$ and $k \geq 0$ |
| (2) $i, j, k \geq 0, i^* \geq j, i < k$          | (8) $i \geq 0, j < 0$ and $k < 0$    |
| (3) $i, j, k \geq 0, i^* < j, k \geq i$          | (9) $i < 0, j \geq 0$ and $k \geq 0$ |
| (4) $i, j, k \geq 0, i^* < j, k < i, j^* \leq k$ | (10) $i < 0, j \geq 0$ and $k < 0$   |
| (5) $i, j, k \geq 0, i^* < j, k < i, j^* > k$    | (11) $i < 0, j < 0$ and $k \geq 0$   |
| (6) $i, j \geq 0$ and $k < 0$                    | (12) $i, j, k < 0$ .                 |

Observe that, if  $x \in [-n, m]$ , then, from Corollary 4.4, we have  $x' = x^*$ , and from Lemma 4.5 we have that  $x'' = x$ ; and in particular  $0'' = 0$ . Thus, we conclude that  $\langle [-n, m]; \Rightarrow, 0 \rangle \in \mathbf{I}_{2,0}$ .  $\square$

In view of the above lemma and Theorem 3.8, the relation defined by

$$x \sqsubseteq y \quad \text{if and only if} \quad (x \Rightarrow y')' = x$$

is a partial order on  $[-\mathbf{n}, \mathbf{m}]$ . We now wish to show that  $\sqsubseteq$  is indeed a total order.

**Lemma 4.8** *Let  $[-\mathbf{n}, \mathbf{m}]$  be the algebra, as defined in Definition 4.2. Then*

$$\langle [-n, m]; \sqsubseteq \rangle \cong \langle [-n, m]; \leq \rangle.$$

**Proof** Let  $x, y \in [-n, m]$ . It is enough to prove that  $x \leq y$  if and only if  $x \sqsubseteq y$ .

Assume that  $x \leq y$ . We will consider the following cases:

- **Case 1:**  $x < 0$ . Then

$$(4.2) \quad (x \Rightarrow y')' = (x \Rightarrow y^*)^* = [\min(x, y^*)]^*.$$

We consider further the following subcases:

- **Case 1.1:**  $y < 0$ .

$$\begin{aligned} (x \Rightarrow y')' &= [\min(x, y^*)]^* && \text{by (4.2)} \\ &= [\min(x, y)]^* && \text{since } y < 0 \\ &= x^* && \text{since } x \leq y \\ &= x. && \text{since } x < 0 \end{aligned}$$

- **Case 1.2:**  $y \geq 0$ .

$$\begin{aligned} (x \Rightarrow y')' &= [\min(x, y^*)]^* && \text{by (4.2)} \\ &= x^* && \text{since } y^* \geq 0 \text{ by Lemma 4.3, and } x < 0 \\ &= x. \end{aligned}$$

- **Case 2:**  $x \geq 0$ . Therefore  $y \geq 0$ . In this case

$$\begin{aligned} (x \Rightarrow y')' &= (x \Rightarrow y^*)^* \\ &= [\max(x^*, y^*)]^* \\ &= x^{**} && \text{by Lemma 4.6} \\ &= x \end{aligned}$$

Thus, in all these cases,  $x \sqsubseteq y$ .

For the converse, suppose  $x \sqsubseteq y$ .

- **Case 1:**  $x < 0$ . If  $y \geq 0$  then  $x < y$ . So, we can assume  $y < 0$ . Then

$$\begin{aligned} x &= x' && \text{since } x < 0 \\ &= (x \Rightarrow y')'' && \text{by hypothesis} \\ &= x \Rightarrow y' && \text{by Lemma 4.5} \\ &= x \Rightarrow y \\ &= \min(x, y). \end{aligned}$$

Hence  $x \leq y$ .

- **Case 2:**  $x \geq 0$ . Suppose  $y < 0$ . Then

$$\begin{aligned}
x &= (x \Rightarrow y')' && \text{by hypothesis} \\
&= (x \Rightarrow y)' \\
&= \min(x, y)' \\
&= y' \\
&= y,
\end{aligned}$$

a contradiction. Hence  $y \geq 0$ . Consequently,

$$\begin{aligned}
x' &= (x \Rightarrow y')'' \\
&= x \Rightarrow y' && \text{by Lemma 4.5} \\
&= \max(x', y'),
\end{aligned}$$

so,  $x' \geq y'$ . Then, by Lemma 4.5 and Lemma 4.6,  $x = x'' \leq y'' = y$ .

□

In view of Lemma 4.7 and Lemma 4.8, we have proved the following

**Theorem 4.9**  $[-n, m]$  is an  $\mathbf{I}_{2,0}$ -chain, where

$$-n \sqsubseteq -n + 1 \sqsubseteq \dots \sqsubseteq -1 \sqsubseteq 0 \sqsubseteq 1 \sqsubseteq 2 \sqsubseteq \dots \sqsubseteq m.$$

## 5 Characterization of finite $\mathbf{I}_{2,0}$ -chains

In this section we are going to prove our second main result. The following lemmas will be useful later in this section.

**Lemma 5.1** Let  $\mathbf{A} \in \mathbf{I}_{2,0}$ . Then  $0'$  is the greatest element in  $A$ , relative to  $\sqsubseteq$ .

**Proof** Let  $a \in A$ . Since  $(a \rightarrow (0 \rightarrow 0)')' = (a \rightarrow 0'')' = (a \rightarrow 0)' = a'' = a$ , we have  $a \sqsubseteq 0'$ . □

**Lemma 5.2** Let  $\mathbf{A} \in \mathbf{I}_{2,0}$  and let  $a, b \in A$  with  $0 \sqsubseteq a \sqsubseteq b$ . Then  $b' \sqsubseteq a'$ .

**Proof**

$$\begin{aligned}
(b' \rightarrow a'')' &= (b' \rightarrow a)' \\
&= (b' \rightarrow (a \rightarrow b')')' && \text{by hypothesis} \\
&= ((a \rightarrow 0') \rightarrow b'')' && \text{by Lemma 2.6 (26)} \\
&= ((a \rightarrow 0') \rightarrow b)' \\
&= ((0 \rightarrow a') \rightarrow b)' && \text{by Lemma 2.5 (a)} \\
&= ((0 \rightarrow a'') \rightarrow b)' \\
&= (0' \rightarrow b)' && \text{by hypothesis} \\
&= b' && \text{by Lemma 2.4 (a)}.
\end{aligned}$$

□

**Lemma 5.3** Let  $\mathbf{A} \in \mathbf{I}_{2,0}$  and let  $a \in A$ . If  $0 \sqsubseteq a$  then  $0 \rightarrow a = 0'$ .

**Proof** First notice that, since  $0 \sqsubseteq a$ ,  $0' = (0 \rightarrow a')'' = 0 \rightarrow a'$ . Consequently,

$$(5.1) \quad 0' = 0 \rightarrow a'.$$

Then

$$\begin{aligned} 0' &= 0' \rightarrow 0' && \text{by Lemma 2.4 (a)} \\ &= (0 \rightarrow a') \rightarrow 0' && \text{by (5.1)} \\ &= (0' \rightarrow a') \rightarrow 0' && \text{by Lemma 2.6 (5)} \\ &= a' \rightarrow 0' && \text{by Lemma 2.4 (a)} \\ &= 0 \rightarrow a. && \text{by Lemma 2.5 (a)} \end{aligned}$$

□

**Lemma 5.4** *Let  $\mathbf{A} \in \mathbf{I}_{2,0}$  and let  $a, b \in A$ . If  $0 \sqsubseteq a$  and  $0 \sqsubseteq b$  then  $0 \sqsubseteq a \rightarrow b$ .*

**Proof**

$$\begin{aligned} [0 \rightarrow (a \rightarrow b)]' &= [(a \rightarrow b) \rightarrow 0']' && \text{by Lemma 2.5 (a)} \\ &= (0 \rightarrow a) \rightarrow (b \rightarrow 0')' && \text{by (I)} \\ &= (0 \rightarrow a) \rightarrow (0 \rightarrow b')' && \text{by Lemma 2.5 (a)} \\ &= (0 \rightarrow a) \rightarrow 0 && \text{since } 0 \sqsubseteq b \\ &= 0' \rightarrow 0 && \text{by Lemma 5.3 since } 0 \sqsubseteq a \\ &= 0. && \text{by Lemma 2.4 (a)} \end{aligned}$$

□

**Corollary 5.5** *Let  $\mathbf{A} \in \mathbf{I}_{2,0}$  and  $a \in A$ . If  $a \sqsupseteq 0$  then  $a' \sqsupseteq 0$ .*

**Lemma 5.6** *Let  $\mathbf{A}$  be an  $\mathbf{I}_{2,0}$ -chain and let  $a, b \in A$ . Then  $a' \rightarrow b' = b \rightarrow a$ .*

**Proof** Since  $\mathbf{A}$  is a chain, we can assume that  $b' \sqsubseteq a$  or  $a \sqsubseteq b'$ .

If  $b' \sqsubseteq a$ ,  $(b' \rightarrow a')' = b'$ , then  $b' \rightarrow a' = b$ . Hence  $b \rightarrow a = (b' \rightarrow a') \rightarrow a = [(a' \rightarrow b') \rightarrow (a' \rightarrow a)']'$ , using (I). By Lemma 2.4 (d),  $[(a' \rightarrow b') \rightarrow (a' \rightarrow a)']' = [(a' \rightarrow b') \rightarrow a']' = [[(a \rightarrow a') \rightarrow (b' \rightarrow a')']']' = (a \rightarrow a') \rightarrow (b' \rightarrow a')' = (a'' \rightarrow a') \rightarrow (b' \rightarrow a')' = a' \rightarrow b'$ .

If  $a \sqsubseteq b'$  then we have  $a' = (a \rightarrow b'')'' = a \rightarrow b$ , and the rest of the argument is similar to the previous case. □

**Lemma 5.7** *Let  $\mathbf{A}$  be a  $\mathbf{I}_{2,0}$ -chain with  $|A| \geq 2$  and let  $a \in A$  such that  $a \sqsubset 0$ . Then*

- (a)  $0 \rightarrow a' = a'$
- (b)  $0 \rightarrow a = a$
- (c)  $(a \rightarrow a) \rightarrow a = a \rightarrow a$
- (d)  $a \rightarrow a = a'$
- (e)  $a \rightarrow a = a$
- (f)  $a = a'$ .

**Proof**

- (a) Since  $a \sqsubseteq 0$ , we have that  $a = (a \rightarrow 0')'$ . Therefore,  $a' = (a \rightarrow 0')'' = a \rightarrow 0' = 0 \rightarrow a'$  by Lemma 2.5 (b).

(b) Since  $a \sqsubseteq 0$ , we have

$$(5.2) \quad a = (a \rightarrow 0')'.$$

Then we get

$$\begin{aligned} (0 \rightarrow a) \rightarrow 0' &= [(0 \rightarrow 0) \rightarrow (a \rightarrow 0')']' && \text{by (I)} \\ &= [(0 \rightarrow 0) \rightarrow a']' && \text{by (5.2)} \\ &= [0' \rightarrow a]' \\ &= a' && \text{by lemma 2.4 (a)} \end{aligned}$$

Using Lemma 2.5 (b), we obtain

$$(5.3) \quad a' = 0 \rightarrow (0 \rightarrow a)'.$$

Since  $\mathbf{A}$  is a chain,  $0 \sqsubseteq 0 \rightarrow a$  or  $0 \rightarrow a \sqsubseteq 0$ . Suppose that  $0 \sqsubseteq 0 \rightarrow a$ . Then  $(0 \rightarrow (0 \rightarrow a)')' = 0$ . Therefore, by (5.3),  $a = a'' = (0 \rightarrow (0 \rightarrow a)')' = 0$ , a contradiction, since  $a \neq 0$ . Consequently,  $0 \rightarrow a \sqsubseteq 0$ . Hence, we have

$$\begin{aligned} 0 \rightarrow a &= ((0 \rightarrow a) \rightarrow 0')' && \text{since } 0 \rightarrow a \sqsubseteq 0 \\ &= (0 \rightarrow (0 \rightarrow a)')' && \text{by lemma 2.5 (b)} \\ &= a'' && \text{by (5.3)} \\ &= a. \end{aligned}$$

(c)

$$\begin{aligned} a \rightarrow a &= (0 \rightarrow a) \rightarrow a && \text{by item (b)} \\ &= (a' \rightarrow 0') \rightarrow a && \text{by lemma 2.5 (a)} \\ &= [(a' \rightarrow a') \rightarrow (0' \rightarrow a)']' && \text{by (I)} \\ &= [(a \rightarrow a) \rightarrow (0' \rightarrow a)']' && \text{by Lemma 5.6} \\ &= [(a \rightarrow a) \rightarrow a']' && \text{by lemma 2.5 (a)} \\ &= [[(a'' \rightarrow a) \rightarrow (a \rightarrow a')']']' && \text{by (I)} \\ &= (a \rightarrow a) \rightarrow (a \rightarrow a')' \\ &= (a \rightarrow a) \rightarrow (a'' \rightarrow a')' \\ &= (a \rightarrow a) \rightarrow a'' && \text{by lemma 2.5 (d)} \\ &= (a \rightarrow a) \rightarrow a. \end{aligned}$$

(d) Since  $\mathbf{A}$  is a chain,  $0 \rightarrow a' \sqsubseteq a$  or  $a \sqsubseteq 0 \rightarrow a'$ .

First, we assume that  $0 \rightarrow a' \sqsubseteq a$ . Then

$$\begin{aligned} a \rightarrow a &= (a \rightarrow a) \rightarrow a && \text{by (c)} \\ &= (a' \rightarrow a') \rightarrow a && \text{by Lemma 5.6} \\ &= a' \rightarrow (a' \rightarrow a')' && \text{by Lemma 5.6} \\ &= (a \rightarrow 0) \rightarrow (a' \rightarrow a')' \\ &= [(a \rightarrow 0) \rightarrow (a' \rightarrow a')']'' \\ &= [(0 \rightarrow a') \rightarrow a']' && \text{using (I)} \\ &= 0 \rightarrow a' && \text{since } 0 \rightarrow a' \sqsubseteq a \\ &= a' && \text{using (a).} \end{aligned}$$

Next, we assume  $a \sqsubseteq 0 \rightarrow a'$ , i.e.,  $(a \rightarrow (0 \rightarrow a')')' = a$ . Then, from (a), we have  $a \rightarrow a = (a \rightarrow a'')'' = [(a \rightarrow (0 \rightarrow a')')']' = a'$ .



(e) Using the items (c), (d) and Lemma 2.5 (d), we have  $a \rightarrow a = (a \rightarrow a) \rightarrow a = a' \rightarrow a = a$ .

(f) This follows immediately from the two preceding items.

□

**Lemma 5.8** *Let  $\mathbf{A}$  be an  $\mathbf{I}_{2,0}$ -chain with  $|A| \geq 2$ , and let  $a, b \in A$ . If  $0 \sqsubseteq a$  and  $b \sqsubset 0$  then  $b \rightarrow a = b$  and  $a \rightarrow b = b$ .*

**Proof** Since  $0 \sqsubseteq a$  and  $b \sqsubset 0$ , we have that  $(0 \rightarrow a')' = 0$  and  $(b \rightarrow 0')' = b$ . Therefore, using Lemma 5.6,  $b = b'' = b' \rightarrow 0 = (b \rightarrow 0')'' \rightarrow (0 \rightarrow a')' = (b \rightarrow 0') \rightarrow (0 \rightarrow a')' = (0 \rightarrow b') \rightarrow (a \rightarrow 0')' = [(b' \rightarrow a) \rightarrow 0']'$ . Hence,

$$(5.4) \quad b = [(b' \rightarrow a) \rightarrow 0']'.$$

From the hypothesis and Lemma 5.7 (f), we have

$$(5.5) \quad b' = b.$$

Suppose that  $0 \sqsubseteq b' \rightarrow a$ . Then  $0 = [0 \rightarrow (b' \rightarrow a)]' = [(b' \rightarrow a) \rightarrow 0']'$  by Lemma 5.6, implying  $0 = b$ , which is a contradiction in view of (5.4). Consequently,  $b' \rightarrow a \sqsubseteq 0$ , since  $\mathbf{A}$  is a chain. Hence,

$$(5.6) \quad b' \rightarrow a = [(b' \rightarrow a) \rightarrow 0']'.$$

From (5.4), (5.5) and (5.6) we conclude  $b = b \rightarrow a$ , proving the first half of the conclusion of the lemma. From

$$\begin{aligned} b &= (b \rightarrow a')' && \text{since } b \sqsubseteq a, \text{ as } 0 \sqsubseteq a \text{ and } b \sqsubset 0 \\ &= (a'' \rightarrow b')' && \text{by Lemma 5.6} \\ &= (a \rightarrow b')' \\ &= (a \rightarrow b)' && \text{by (5.5)} \end{aligned}$$

we conclude that  $a \rightarrow b = b' = b$  in view of (5.5), completing the second half. □

**Definition 5.9** *Let  $\mathbf{A} = \langle A; \rightarrow, 0 \rangle$  be a finite  $\mathbf{I}_{2,0}$ -chain. We let  $A^+ := \{a \in A : a \sqsubset 0\}$  and  $A^- := \{a \in A : a \sqsubset 0\}$ . Observe that  $A = A^+ \cup \{0\} \cup A^-$ . Henceforth, without loss of generality, we will represent  $A = [-n, m]$  with  $0 \leq n, m \leq |A| - 1$ , such that*

$$-n \sqsubset -n+1 \sqsubset \dots \sqsubset -1 \sqsubset 0 \sqsubset 1 \sqsubset 2 \sqsubset \dots \sqsubset m.$$

**Remark 5.10** *In view of the above definition, we can use the functions  $*$  and  $p$  of Definition 4.2 as functions on the domain  $[-n, m]$  of  $\mathbf{A}$  as well.*

Now, we wish to prove that  $\langle A; \rightarrow, 0 \rangle = \langle [-n, m]; \Rightarrow, 0 \rangle$ . To achieve this, we need the following lemmas.

**Lemma 5.11** *Let  $\mathbf{A} = \langle A; \rightarrow, 0 \rangle$  be a finite  $\mathbf{I}_{2,0}$ -chain with  $|A| \geq 2$ . If  $a \sqsubset 0$  then  $a' = p(p(a)')$ .*

**Proof** By hypothesis we have that  $a \sqsubset 0$ . Then  $p(a) \sqsupseteq 0$ . Hence  $0 \sqsubseteq p(a) \sqsubset a$ . Then, by Lemma 5.2,

$$(5.7) \quad a' \sqsubseteq p(a)'.$$

Since  $a \sqsubset 0$ , by Corollary 5.5,  $a' \sqsupseteq 0$ . Therefore, by (5.7),

$$(5.8) \quad 0 \sqsubseteq p(a)'.$$

If  $a' = p(a)'$  then  $a = p(a)$  and, consequently,  $a = -n$ , a contradiction, so  $a' \sqsubset p(a)'$ , and hence,  $0 \sqsubseteq a' \sqsubseteq p(p(a'))' \sqsubset p(a)'$ . By lemma 5.2,  $a \sqsupseteq [p(p(a'))]' \sqsupseteq p(a)$ . Thus

$$(5.9) \quad [p(p(a'))]' \in \{a, p(a)\}.$$

If  $[p(p(a'))]' = p(a)$ , we have that  $p(p(a')) = [p(p(a'))]'' = p(a)'$ , a contradiction, since  $p(a)' \sqsupseteq 0$  by (5.8). Therefore  $[p(p(a'))]' = a$  and therefore,  $p(p(a')) = a'$ .  $\square$

**Lemma 5.12** *Let  $\mathbf{A} = \langle A; \rightarrow, 0 \rangle$  be a finite  $\mathbf{I}_{2,0}$ -chain. If  $a \in A$  then  $a^* = a'$ .*

**Proof** The statement  $0' = m = 0^*$  follows from Lemma 5.1. If  $a \sqsubset 0$  then  $a' = a$  by Lemma 5.7 (f), and  $a = a^*$  by definition, implying  $a = a^*$ .

Now assume that  $a \sqsupset 0$ . We will verify that  $a' = a^*$  by induction on  $a$ . If  $a = 1$ , then, as  $0' = 0^*$ , we have, by Lemma 5.11, that  $1' = p(p(1')) = p(0') = p(0^*) = p(p(1)^*) = 1^*$ . The inductive hypothesis is that  $p(a)' = p(a)^*$ . Hence, we have, by Lemma 5.11,  $a' = p(p(a')) = p(p(a)^*) = a^*$ .  $\square$

The following theorem shows that the general method described in Definition 4.2 essentially gives all finite  $\mathbf{I}_{2,0}$ -chains.

**Theorem 5.13** *Let  $\mathbf{A}$  be a finite  $\mathbf{I}_{2,0}$ -chain. Then  $\mathbf{A} \cong \langle [-n, m]; \Rightarrow, 0 \rangle$  for some  $0 \leq n, m \leq |A| - 1$ .*

**Proof** We will use the notation of Definition 5.9. Let  $i, j \in A$ . From Lemma 5.12,  $i' = i^*$  and  $j' = j^*$ . It suffices to verify that

$$i \rightarrow j = \begin{cases} \max(i', j) & \text{if } i, j \sqsupseteq 0 \\ \min(i, j) & \text{otherwise} \end{cases}$$

with  $0' = m$ . We consider the following cases:

- **Case 1:**  $j > 0$ .

We need the following subcases:

- **Case 1.1:**  $i > 0$ .

We make the following further subcases:

- \* **Case 1.1.1:**  $i' \geq j$ .

Since  $i' \sqsupseteq j$ , we observe that

$$(5.10) \quad (j \rightarrow i'')' = j.$$

Hence

$$\begin{aligned} i \rightarrow j &= i \rightarrow (j \rightarrow i'')' && \text{by (5.10)} \\ &= i \rightarrow (j \rightarrow i)' \\ &= [(i \rightarrow j) \rightarrow i]' && \text{by Lemma 2.6 (4)} \\ &= [(0 \rightarrow j) \rightarrow i]' && \text{by Lemma 2.6 (5)} \\ &= [0' \rightarrow i]' && \text{by Lemma 5.3 since } j \sqsupseteq 0 \\ &= i' && \text{by Lemma 2.4 (a)} \\ &= \max(i', j) && \text{since } i' \sqsupseteq j \end{aligned}$$

\* **Case 1.1.2:**  $i' < j$ .

Since  $i' \sqsubseteq j$ , we have

$$(5.11) \quad (i' \rightarrow j')' = i'.$$

Therefore,

$$\begin{aligned} i \rightarrow j &= i'' \rightarrow j \\ &= (i' \rightarrow j')'' \rightarrow j && \text{by (5.11)} \\ &= (i' \rightarrow j') \rightarrow j \\ &= (i' \rightarrow 0') \rightarrow j && \text{by Lemma 2.6 (1)} \\ &= (0 \rightarrow i) \rightarrow j && \text{by Lemma 2.5 (a)} \\ &= 0' \rightarrow j && \text{by Lemma 5.3 since } i \sqsupseteq 0 \\ &= j && \text{by Lemma 2.4 (a)} \\ &= \max(i', j) && \text{since } i' \sqsubseteq j \end{aligned}$$

– **Case 1.2:**  $i = 0$ .

Using Lemma 5.3 and Lemma 5.1,  $0 \rightarrow j = 0' = \max(0', j)$ .

– **Case 1.3:**  $i < 0$ .

$$\begin{aligned} i \rightarrow j &= (0 \rightarrow i) \rightarrow j && \text{by Lemma 5.7 (b)} \\ &= (i' \rightarrow 0') \rightarrow j \\ &= (i \rightarrow 0') \rightarrow j && \text{by Lemma 5.7 (f)} \\ &= [(j' \rightarrow i) \rightarrow (0' \rightarrow j)]' && \text{by (I)} \\ &= [(j' \rightarrow i) \rightarrow j']' \\ &= [(0 \rightarrow i) \rightarrow j']' && \text{by Lemma 2.6 (5)} \\ &= (i \rightarrow j')' && \text{by Lemma 5.7 (b)} \\ &= i && \text{since } i \sqsubset j \\ &= \min(i, j) \end{aligned}$$

• **Case 2:**  $j < 0$ .

It is useful to consider the following subcases:

– **Case 2.1:**  $i > 0$

$$\begin{aligned} i \rightarrow j &= i \rightarrow j' && \text{by Lemma 5.7 (f)} \\ &= i \rightarrow (j \rightarrow i')'' && \text{since } j \sqsubset i \\ &= i \rightarrow (j \rightarrow i') \\ &= j \rightarrow i' && \text{by Lemma 2.6 (17)} \\ &= (j \rightarrow i')'' \\ &= j' && \text{since } j \sqsubset i \\ &= j && \text{by Lemma 5.7 (f)} \\ &= \min(i, j) \end{aligned}$$

– **Case 2.2:**  $i = 0$ .

$$\begin{aligned} i \rightarrow j &= 0 \rightarrow j \\ &= j && \text{by Lemma 5.7 (b)} \\ &= \min(i, j) \end{aligned}$$

– **Case 2.3:**  $i < 0$ .

\* **Case 2.3.1:**  $i \leq j$ .

As  $i \sqsubseteq j$ , we have

$$(5.12) \quad (i \rightarrow j')' = i.$$

Observe

$$\begin{aligned} i \rightarrow j &= i \rightarrow j' && \text{by Lemma 5.7 (f)} \\ &= (i \rightarrow j')'' \\ &= i' && \text{by (5.12)} \\ &= i && \text{by Lemma 5.7 (f)} \\ &= \min(i, j). \end{aligned}$$

\* **Case 2.3.2:**  $i > j$ . We have

$$(5.13) \quad (j \rightarrow i')' = j.$$

as  $j \sqsubseteq i$ . Hence

$$\begin{aligned} i \rightarrow j &= j' \rightarrow i' && \text{by Lemma 5.6} \\ &= j \rightarrow i' && \text{by Lemma 5.7 (f)} \\ &= (j \rightarrow i')'' \\ &= j' && \text{by (5.13)} \\ &= j && \text{by Lemma 5.7 (f)} \\ &= \min(j, i) \end{aligned}$$

• **Case 3:**  $j = 0$ .

– **Case 3.1:**  $i \geq 0$ .

By Corollary 5.5, as  $i \sqsupseteq 0$ , we have that  $i' = i \rightarrow 0 \sqsupseteq 0$ . Hence  $i \rightarrow 0 = i' = \max(i', 0)$ .

– **Case 3.2:**  $i < 0$ . We have that

$$\begin{aligned} i \rightarrow j &= i \rightarrow 0 \\ &= i' \\ &= i && \text{by Lemma 5.7 (f)} \\ &= \min(i, j) \end{aligned}$$

Hence  $\mathbf{A} \cong \langle [-n; m]; \Rightarrow, 0 \rangle$ . □

The following theorem, our second main result, is now immediate from the preceding results.

**Theorem 5.14** *There are  $n$  non-isomorphic  $I_{2,0}$ -chains of size  $n$ , for  $n \in \mathbb{N}$ .*

## A Appendix: Proofs

We would like to mention here that the identity:  $x'' \approx x$  is used in these proofs frequently without explicit mention.

**Proof of Lemma 2.6:** Items (1) to (17) are proved in [3]. The proofs of (18) to (26) are given in [5]. Let  $a, b, c, d \in A$ .

(27)

$$\begin{aligned}
(b \rightarrow c) \rightarrow a &= [(b \rightarrow c) \rightarrow a]' \rightarrow [(b \rightarrow c) \rightarrow a] && \text{by Lemma 2.4 (d)} \\
&= [(a' \rightarrow b) \rightarrow (c \rightarrow a)']'' \rightarrow [(b \rightarrow c) \rightarrow a] && \text{from (I)} \\
&= [(a' \rightarrow b) \rightarrow (c \rightarrow a)'] \rightarrow [(b \rightarrow c) \rightarrow a]
\end{aligned}$$

(28)

$$\begin{aligned}
[[0 \rightarrow (a \rightarrow b)'] \rightarrow (0 \rightarrow b')']' &= [\{0 \rightarrow (a \rightarrow b)'\} \rightarrow (b \rightarrow 0')']' && \text{by Lemma 2.5 (a)} \\
&= [(a \rightarrow b)' \rightarrow b] \rightarrow 0' && \text{by (I)} \\
&= 0 \rightarrow [(a \rightarrow b)' \rightarrow b]' && \text{by Lemma 2.5 (a)} \\
&= (a \rightarrow b) \rightarrow (0 \rightarrow b') && \text{by (10)} \\
&= 0 \rightarrow [(a \rightarrow b) \rightarrow b'] && \text{by (13)} \\
&= 0 \rightarrow [(a \rightarrow 0') \rightarrow b'] && \text{by (1)} \\
&= (a \rightarrow 0') \rightarrow (0 \rightarrow b') && \text{by (13)} \\
&= (0 \rightarrow a') \rightarrow (0 \rightarrow b') && \text{by Lemma 2.5 (a)} \\
&= a' \rightarrow (0 \rightarrow b') && \text{by (20)} \\
&= [(0 \rightarrow a) \rightarrow (0 \rightarrow b')']' && \text{by (8)} \\
&= [(0 \rightarrow a) \rightarrow (b \rightarrow 0')']' && \text{by Lemma 2.5 (a)} \\
&= (a \rightarrow b) \rightarrow 0' && \text{by (I)} \\
&= 0 \rightarrow (a \rightarrow b)' && \text{by Lemma 2.5 (a)}
\end{aligned}$$

(29)

$$\begin{aligned}
0 \rightarrow [(a \rightarrow b) \rightarrow c] &= 0 \rightarrow [(c' \rightarrow a) \rightarrow (b \rightarrow c)']' && \text{by (I)} \\
&\sqsubseteq 0 \rightarrow (b \rightarrow c)'' && \text{by (28)} \\
&= 0 \rightarrow (b \rightarrow c)
\end{aligned}$$

(30)

$$\begin{aligned}
a' \rightarrow (b \rightarrow 0')' &= (a \rightarrow 0) \rightarrow (b \rightarrow 0')' \\
&= [\{(b \rightarrow 0') \rightarrow a\} \rightarrow \{0 \rightarrow (b \rightarrow 0')'\}]' && \text{by (I)} \\
&= [\{(b \rightarrow 0') \rightarrow a\} \rightarrow \{0 \rightarrow (0 \rightarrow b')'\}]' && \text{by Lemma 2.5 (a)} \\
&= [\{(b \rightarrow 0') \rightarrow a\} \rightarrow (0 \rightarrow b)']' && \text{by (9)} \\
&= [\{(0 \rightarrow b') \rightarrow a\} \rightarrow (0 \rightarrow b)']' && \text{by Lemma 2.5 (a)} \\
&= [[\{0 \rightarrow (0 \rightarrow b')\} \rightarrow a] \rightarrow (0 \rightarrow b)']' && \text{by (9)} \\
&= [a \rightarrow (0 \rightarrow b)']' && \text{by (18)}
\end{aligned}$$

(31)

$$\begin{aligned}
[(0 \rightarrow a) \rightarrow b]' &= [(b \rightarrow a) \rightarrow b]' && \text{by (5)} \\
&= [b \rightarrow (a \rightarrow b)']'' && \text{by (4)} \\
&= b \rightarrow (a \rightarrow b)'
\end{aligned}$$

(32)

$$\begin{aligned}
[a \rightarrow (b \rightarrow 0')']' &= [a \rightarrow (0 \rightarrow b')']' && \text{by Lemma 2.5 (a)} \\
&= a' \rightarrow (b' \rightarrow 0')' && \text{by (30)} \\
&= a' \rightarrow (0 \rightarrow b)' && \text{by Lemma 2.5 (a)}
\end{aligned}$$

(33)

$$\begin{aligned}
b' \rightarrow a' &= (b \rightarrow 0) \rightarrow a' \\
&= [(a \rightarrow b) \rightarrow (0 \rightarrow a')']' && \text{by (I)} \\
&= [(a \rightarrow b) \rightarrow (a \rightarrow 0')']' && \text{by Lemma 2.5 (a)} \\
&= (a \rightarrow b)' \rightarrow (0 \rightarrow a)' && \text{by (32) with } x = a \rightarrow b, y = a
\end{aligned}$$

(34)

$$\begin{aligned}
(0 \rightarrow a)' \rightarrow (0 \rightarrow b)' &= [(0 \rightarrow a) \rightarrow 0] \rightarrow (0 \rightarrow b)' \\
&= [\{(0 \rightarrow b) \rightarrow (0 \rightarrow a)\} \rightarrow \{0 \rightarrow (0 \rightarrow b)'\}]' \\
&\quad \text{by (I)} \\
&= [\{0 \rightarrow (0 \rightarrow b)'\} \rightarrow (0 \rightarrow b)] \rightarrow [(0 \rightarrow a) \rightarrow \{0 \rightarrow (0 \rightarrow b)'\}]' \\
&\quad \text{by (I)} \\
&= [\{(0 \rightarrow b) \rightarrow (0 \rightarrow b)'\} \rightarrow (0 \rightarrow b)] \rightarrow [(0 \rightarrow a) \rightarrow \{0 \rightarrow (0 \rightarrow b)'\}]' \\
&\quad \text{by (5)} \\
&= [(0 \rightarrow b)' \rightarrow (0 \rightarrow b)] \rightarrow [(0 \rightarrow a) \rightarrow \{0 \rightarrow (0 \rightarrow b)'\}]' \\
&\quad \text{by Lemma 2.4 (d)} \\
&= (0 \rightarrow b) \rightarrow [(0 \rightarrow a) \rightarrow \{0 \rightarrow (0 \rightarrow b)'\}]' \\
&\quad \text{by Lemma 2.4 (d)} \\
&= (0 \rightarrow b) \rightarrow [(0 \rightarrow a) \rightarrow (0 \rightarrow b')]' \\
&\quad \text{by (9)} \\
&= (0 \rightarrow b) \rightarrow [(0 \rightarrow a) \rightarrow (b \rightarrow 0')]' \\
&\quad \text{by Lemma 2.5 (a)} \\
&= (0 \rightarrow b) \rightarrow [(a \rightarrow b) \rightarrow 0'] \\
&\quad \text{by (I)} \\
&= (0 \rightarrow b) \rightarrow [0 \rightarrow (a \rightarrow b)'] \\
&\quad \text{by Lemma 2.5 (a)} \\
&= 0 \rightarrow [(0 \rightarrow b) \rightarrow (a \rightarrow b)'] \\
&\quad \text{by (13)} \\
&= 0 \rightarrow (a \rightarrow b)' \\
&\quad \text{by (3)} \\
&= (a \rightarrow b) \rightarrow 0' \\
&\quad \text{by Lemma 2.5 (a)} \\
&= [(0 \rightarrow a) \rightarrow (b \rightarrow 0')]' \\
&\quad \text{by (I)} \\
&= [(0 \rightarrow a) \rightarrow (0 \rightarrow b')]' \\
&\quad \text{by Lemma 2.5 (a)} \\
&= [a' \rightarrow (0 \rightarrow b')]'' \\
&\quad \text{by (8)} \\
&= a' \rightarrow (0 \rightarrow b') \\
&= 0 \rightarrow (a' \rightarrow b') \quad \text{by (13).}
\end{aligned}$$

(35)

$$\begin{aligned}
[(a \rightarrow b)' \rightarrow \{b \rightarrow (a \rightarrow b)'\}]' &= [(a \rightarrow b)' \rightarrow b] \rightarrow (a \rightarrow b)' \quad \text{by (4)} \\
&= (0 \rightarrow b) \rightarrow (a \rightarrow b)' \quad \text{by (5)} \\
&= (a \rightarrow b)' \quad \text{by (3)}
\end{aligned}$$

(36)

$$\begin{aligned}
(0 \rightarrow a) \rightarrow b &= (b \rightarrow a) \rightarrow b \quad \text{by (5)} \\
&= [b \rightarrow (a \rightarrow b)]' \quad \text{by (4)} \\
&\sqsubseteq (a \rightarrow b)' \rightarrow [b \rightarrow (a \rightarrow b)]' \quad \text{by (35) with } x = b, y = (a \rightarrow b)' \\
&= [\{(a \rightarrow b)' \rightarrow b\} \rightarrow (a \rightarrow b)]' \quad \text{by (4)} \\
&= [(0 \rightarrow b) \rightarrow (a \rightarrow b)]' \quad \text{by (5)} \\
&= (a \rightarrow b)'' \quad \text{by (3)} \\
&= a \rightarrow b \quad \text{since } x'' \approx x
\end{aligned}$$

(37)

$$\begin{aligned}
[\{a \rightarrow (b \rightarrow a)'\} \rightarrow a'']' &= [\{a \rightarrow (b \rightarrow a)'\} \rightarrow a]' \\
&= [\{0 \rightarrow (b \rightarrow a)'\} \rightarrow a]' && \text{by (5)} \\
&= [\{(b \rightarrow a) \rightarrow 0'\} \rightarrow a]' && \text{by Lemma 2.5 (a)} \\
&= [\{(b \rightarrow a) \rightarrow a'\} \rightarrow a]' && \text{by (1)} \\
&= [\{(b \rightarrow 0') \rightarrow a'\} \rightarrow a]' && \text{by (1)} \\
&= [\{(b \rightarrow 0') \rightarrow 0'\} \rightarrow a]' && \text{by (1)} \\
&= [\{(b \rightarrow 0'') \rightarrow 0'\} \rightarrow a]' && \text{by (1)} \\
&= [\{(b \rightarrow 0) \rightarrow 0'\} \rightarrow a]' \\
&= [(b' \rightarrow 0') \rightarrow a]' \\
&= [(0 \rightarrow b) \rightarrow a]' && \text{by Lemma 2.5 (a)} \\
&= [(a \rightarrow b) \rightarrow a]' && \text{by (5)} \\
&= a \rightarrow (b \rightarrow a)' && \text{by (4)}
\end{aligned}$$

□

**Proof of Lemma 3.3**

(1) Observe that by Lemma 2.5 (a), Lemma 2.6 (1) and the hypothesis we have that  $(0 \rightarrow a') \rightarrow b = (a \rightarrow 0') \rightarrow b = (a \rightarrow b') \rightarrow b = (a \rightarrow b'') \rightarrow b = a' \rightarrow b$ .

(2)

$$\begin{aligned}
b \rightarrow a' &= [(0 \rightarrow a') \rightarrow b] \rightarrow a' && \text{by Lemma 2.6 (18)} \\
&= (a' \rightarrow b) \rightarrow a' && \text{from (1)} \\
&= (0 \rightarrow b) \rightarrow a' && \text{by Lemma 2.6 (5)}.
\end{aligned}$$

(3)

$$\begin{aligned}
b \rightarrow a' &= (0 \rightarrow b) \rightarrow a' && \text{from (2)} \\
&= (0 \rightarrow b) \rightarrow (a \rightarrow b'') && \text{by hypothesis} \\
&= (0 \rightarrow b) \rightarrow (a \rightarrow b') \\
&= (0 \rightarrow b'') \rightarrow (a \rightarrow b') \\
&= a \rightarrow b' && \text{by Lemma 2.6 (2)} \\
&= (a \rightarrow b'') && \\
&= a' && \text{by hypothesis}
\end{aligned}$$

(4)

$$\begin{aligned}
0 \rightarrow (a' \rightarrow b) &= a' \rightarrow (0 \rightarrow b) && \text{by Lemma 2.6 (13)} \\
&= 0 \rightarrow (a \rightarrow b')' && \text{by Lemma 2.6 (10)} \\
&= 0 \rightarrow a && \text{by hypothesis}
\end{aligned}$$

(5) By hypothesis and (I) we have that  $(d \rightarrow a) \rightarrow b' = [(b \rightarrow d) \rightarrow (a \rightarrow b')']' = [(b \rightarrow d) \rightarrow a]'$ .

(6)

$$\begin{aligned}
[\{d \rightarrow (0 \rightarrow b')\} \rightarrow a]' &= (a' \rightarrow d) \rightarrow [(0 \rightarrow b') \rightarrow a]' && \text{by (I)} \\
&= (a' \rightarrow d) \rightarrow [(a \rightarrow b') \rightarrow a]' && \text{by Lemma 2.6 (5)} \\
&= (a' \rightarrow d) \rightarrow [(a \rightarrow b'') \rightarrow a]' \\
&= (a' \rightarrow d) \rightarrow (a' \rightarrow a)' && \text{by hypothesis} \\
&= (a' \rightarrow d) \rightarrow a' && \text{by Lemma 2.4 (d)} \\
&= (a' \rightarrow d) \rightarrow (0' \rightarrow a)' && \text{by Lemma 2.4 (a)} \\
&= [(d \rightarrow 0') \rightarrow a]' && \text{by (I)} \\
&= (0 \rightarrow d) \rightarrow a' && \text{by Lemma 2.6 (11)}
\end{aligned}$$

(7)

$$\begin{aligned}
a \rightarrow [(a' \rightarrow d) \rightarrow \{(0 \rightarrow a) \rightarrow b'\}] &= a \rightarrow [(a' \rightarrow d) \rightarrow \{(b \rightarrow 0) \rightarrow (a \rightarrow b')'\}] \\
&\text{by (I)} \\
&= a \rightarrow [(a' \rightarrow d) \rightarrow \{(b \rightarrow 0) \rightarrow a\}'] \\
&\text{by hypothesis} \\
&= a \rightarrow [\{d \rightarrow (b \rightarrow 0)\} \rightarrow a]' \\
&\text{by (I)} \\
&= [[d \rightarrow (0 \rightarrow (b \rightarrow 0))]] \rightarrow a]' \\
&\text{by Lemma 2.6 (22) with } x = d, y = b \rightarrow 0, z = a \\
&= [[d \rightarrow (0 \rightarrow b')] \rightarrow a]' \\
&= (0 \rightarrow d) \rightarrow a' \\
&\text{by (6)}
\end{aligned}$$

(8)

$$\begin{aligned}
a \rightarrow ((d \rightarrow a) \rightarrow b') &= a \rightarrow [(b \rightarrow d) \rightarrow a]' && \text{by (5)} \\
&= a'' \rightarrow [(b \rightarrow d) \rightarrow a]' \\
&= (a' \rightarrow 0) \rightarrow [(b \rightarrow d) \rightarrow a]' \\
&= [\{0 \rightarrow (b \rightarrow d)\} \rightarrow a]' && \text{by (I)} \\
&= [\{(b \rightarrow d)' \rightarrow 0'\} \rightarrow a]' \\
&= [\{((b \rightarrow d) \rightarrow 0) \rightarrow 0'\} \rightarrow a]' \\
&= [\{((b \rightarrow d) \rightarrow 0') \rightarrow 0'\} \rightarrow a]' && \text{by Lemma 2.6 (1)} \\
&= [\{((b \rightarrow d) \rightarrow 0') \rightarrow a'\} \rightarrow a]' && \text{by Lemma 2.6 (1)} \\
&= [\{((b \rightarrow d) \rightarrow a) \rightarrow a'\} \rightarrow a]' && \text{by Lemma 2.6 (1)} \\
&= [\{((b \rightarrow d) \rightarrow a) \rightarrow 0'\} \rightarrow a]' && \text{by Lemma 2.6 (1)} \\
&= [\{0 \rightarrow ((b \rightarrow d) \rightarrow a)'\} \rightarrow a]' \\
&= [\{0 \rightarrow ((d \rightarrow a) \rightarrow b')\} \rightarrow a]' && \text{by (5)} \\
&= [\{a \rightarrow ((d \rightarrow a) \rightarrow b')\} \rightarrow a]' && \text{by Lemma 2.6 (5)} \\
&= (a' \rightarrow a) \rightarrow [\{(d \rightarrow a) \rightarrow b'\} \rightarrow a]' && \text{by (I)} \\
&= a \rightarrow [\{(d \rightarrow a) \rightarrow b'\} \rightarrow a]' && \text{by Lemma 2.4 (d)} \\
&= a \rightarrow [\{a' \rightarrow (d \rightarrow a)\} \rightarrow (b' \rightarrow a)'] && \text{by (I)} \\
&= a \rightarrow [\{a' \rightarrow (d \rightarrow a)\} \rightarrow \{(b \rightarrow 0) \rightarrow a\}'] \\
&= a \rightarrow [\{a' \rightarrow (d \rightarrow a)\} \rightarrow \{(b \rightarrow 0) \rightarrow (a \rightarrow b')'\}] && \text{by hypothesis} \\
&= a \rightarrow [\{a' \rightarrow (d \rightarrow a)\} \rightarrow \{(0 \rightarrow a) \rightarrow b'\}] && \text{by (I)} \\
&= [0 \rightarrow (d \rightarrow a)] \rightarrow a' && \text{by (7) with } d := d \rightarrow a \\
&= a \rightarrow (d \rightarrow a)' && \text{by Lemma 2.6 (23)}
\end{aligned}$$

(9)

$$\begin{aligned}
[0 \rightarrow (b \rightarrow d)] \rightarrow a &= [(a' \rightarrow 0) \rightarrow ((b \rightarrow d) \rightarrow a)']' && \text{by (I)} \\
&= [a \rightarrow ((b \rightarrow d) \rightarrow a)']' \\
&= [a \rightarrow ((d \rightarrow a) \rightarrow b')] && \text{by (5)} \\
&= [a \rightarrow (d \rightarrow a)'] && \text{by (8)} \\
&= (a \rightarrow d) \rightarrow a && \text{by Lemma 2.6 (4)} \\
&= (0 \rightarrow d) \rightarrow a && \text{by Lemma 2.6 (5)}
\end{aligned}$$



(10)

$$\begin{aligned}
(b \rightarrow (a \rightarrow d)) \rightarrow a &= [(a' \rightarrow b) \rightarrow \{(a \rightarrow d) \rightarrow a\}']' && \text{by (I)} \\
&= [(a' \rightarrow b) \rightarrow \{(0 \rightarrow d) \rightarrow a\}']' && \text{by Lemma 2.6 (5)} \\
&= [b \rightarrow (0 \rightarrow d)] \rightarrow a && \text{by (I)} \\
&= [0 \rightarrow (b \rightarrow d)] \rightarrow a && \text{by Lemma 2.6 (13)} \\
&= (0 \rightarrow d) \rightarrow a && \text{by (9)}
\end{aligned}$$

(11)

$$\begin{aligned}
b \rightarrow (0 \rightarrow a') &= (0 \rightarrow b) \rightarrow (0 \rightarrow a') && \text{by Lemma 2.6 (20)} \\
&= 0 \rightarrow [(0 \rightarrow b) \rightarrow a'] && \text{by Lemma 2.6 (13)} \\
&= 0 \rightarrow [(a \rightarrow 0) \rightarrow (b \rightarrow a')']' && \text{by (I)} \\
&= 0 \rightarrow [a' \rightarrow (b \rightarrow a')']' \\
&= 0 \rightarrow (a' \rightarrow a'')' && \text{by (3)} \\
&= 0 \rightarrow (a' \rightarrow a)' \\
&= 0 \rightarrow a' && \text{by Lemma 2.4 (d)}
\end{aligned}$$

(12) From (I) and by hypothesis we have that  $[(d \rightarrow a) \rightarrow b']' = (b \rightarrow d) \rightarrow (a \rightarrow b')' = (b \rightarrow d) \rightarrow a$ .

(13)

$$\begin{aligned}
a' \rightarrow b &= (a \rightarrow b') \rightarrow b && \text{by hypothesis} \\
&= [(b' \rightarrow a) \rightarrow (b' \rightarrow b)']' && \text{using (I)} \\
&= [(b' \rightarrow a) \rightarrow b']' && \text{by Lemma 2.4 (d)} \\
&= b' \rightarrow (a \rightarrow b')' && \text{by Lemma 2.6 (4)} \\
&= b' \rightarrow a && \text{by hypothesis}
\end{aligned}$$

(14)

$$\begin{aligned}
(d \rightarrow 0') \rightarrow (a' \rightarrow b) &= (a' \rightarrow b)' \rightarrow [(d \rightarrow 0') \rightarrow (a' \rightarrow b)] \\
&\quad \text{by Lemma 2.6 (17)} \\
&= (a' \rightarrow b)' \rightarrow [(d \rightarrow 0') \rightarrow (b' \rightarrow a)] \\
&\quad \text{by (13)} \\
&= (a' \rightarrow b)' \rightarrow [(d \rightarrow 0') \rightarrow \{(0 \rightarrow a) \rightarrow b'\}'] \\
&\quad \text{by (12) with } d = 0 \\
&= (a' \rightarrow b)' \rightarrow [\{(0 \rightarrow a) \rightarrow b'\} \rightarrow \{d \rightarrow ((0 \rightarrow a) \rightarrow b')\}'] \\
&\quad \text{by Lemma 2.6 (26) with} \\
&\quad \quad x = (0 \rightarrow a) \rightarrow b', y = d \\
&= (a' \rightarrow b)' \rightarrow [(b' \rightarrow a)' \rightarrow \{d \rightarrow ((0 \rightarrow a) \rightarrow b')\}'] \\
&\quad \text{by (12) with } d = 0 \\
&= (a' \rightarrow b)' \rightarrow [(a' \rightarrow b)' \rightarrow \{d \rightarrow ((0 \rightarrow a) \rightarrow b')\}'] \\
&\quad \text{by (13)} \\
&= (a' \rightarrow b)' \rightarrow [d \rightarrow \{(0 \rightarrow a) \rightarrow b'\}]' \\
&\quad \text{by Lemma 2.6 (21)} \\
&= (b' \rightarrow a)' \rightarrow [d \rightarrow \{(0 \rightarrow a) \rightarrow b'\}]' \\
&\quad \text{by (13)} \\
&= [(0 \rightarrow a) \rightarrow b'] \rightarrow [d \rightarrow \{(0 \rightarrow a) \rightarrow b'\}]' \\
&\quad \text{by (12) with } d = 0 \\
&= [(0 \rightarrow d) \rightarrow \{(0 \rightarrow a) \rightarrow b'\}]' \\
&\quad \text{by Lemma 2.6 (31) with} \\
&\quad \quad x = d, y = (0 \rightarrow a) \rightarrow b' \\
&= [(b' \rightarrow d) \rightarrow \{(0 \rightarrow a) \rightarrow b'\}]' \\
&\quad \text{by Lemma 2.6 (15) with} \\
&\quad \quad x = b, y = d, z = a \\
&= [(b' \rightarrow d) \rightarrow (b' \rightarrow a)]' \\
&\quad \text{by (12) with } d = 0 \\
&= [(b' \rightarrow d) \rightarrow (a' \rightarrow b)]' \\
&\quad \text{by (13)} \\
&= (d \rightarrow a') \rightarrow b \quad \text{by (I).}
\end{aligned}$$

(15)

$$\begin{aligned}
[(0 \rightarrow a') \rightarrow b]' &= [(a \rightarrow 0') \rightarrow b]' \quad \text{by Lemma 2.5 (a)} \\
&= (0 \rightarrow a) \rightarrow b' \quad \text{by Lemma 2.6 (11)}
\end{aligned}$$

(16)

$$\begin{aligned}
(a' \rightarrow b)' &= [(0 \rightarrow a') \rightarrow b]' \quad \text{by (1)} \\
&= (0 \rightarrow a) \rightarrow b' \quad \text{by (15)}
\end{aligned}$$

(17)

$$\begin{aligned} [\{b' \rightarrow ((b \rightarrow d) \rightarrow a)\} \rightarrow (0 \rightarrow b)']' &= [\{b' \rightarrow ((d \rightarrow a) \rightarrow b')'\} \rightarrow (0 \rightarrow b)']' \\ &\quad \text{by (12)} \\ &= [\{(d \rightarrow a) \rightarrow b'\}' \rightarrow 0] \rightarrow b \\ &\quad \text{by (I)} \\ &= [(d \rightarrow a) \rightarrow b'] \rightarrow b \\ &= b' \rightarrow [(d \rightarrow a) \rightarrow b']' \\ &\quad \text{by Lemma 2.6 (14) with } x = d \rightarrow a, y = b' \\ &= b' \rightarrow [(b \rightarrow d) \rightarrow a] \quad \text{by (12)}. \end{aligned}$$

□

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